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AXIOMATIC THEORY OF CONTROL SYSTEMS

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by

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Axiomatic Theory of Control Systems.

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1. Introduction.

Control systems, in the broadest sense, are those systems whose behaviour is not uniquely determined by the initial conditions and the characteristics of the system itself, but there is some external action which determines the evolution of the system. When there is no external action, one generally says that the system is in its "free regime, but it is really arbitrary, to some extent, where the zero for the external action has to be set. The classical description of the free physical systems is made by differential equations which, together with some initial conditions, determine uniquely its time-evolution. Accordingly, the description of control systems is made mostly in terms of differential equations, in which some "control parameter" appears, so that in order to obtain a solution it is necessary to know the value of that parameter, as a function of time or of position. Before fixing that control function, the only knowledge about the behaviour of the system

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is the whole set of possible states which can be attained by all possible choices of the control action. This set of attainable (or reachable) states has been used recently in many papers about control systems [2] [5] [9]]10] [11].

As a generalization of the theory of differential equations, the topological dynamics studies all properties which are characteristic for solutions of differential equations, without a direct reference to those equations, by assuming as axioms the basic properties of such curves [3] [4] [8]. Similarly, for control systems the basic properties of the solutions can be assumed in order to develop the whole theory in an axiomatic way. Here the axioms have to refer to the properties of the attainable set, this is the set of all attainable states (by means of all possible control actions) as functions of time and initial position. Barbashin [1] gave such axioms for "generalized dynamical systems" and Zubov [13] also used similar general systems for proving theorems about stability. Nevertheless, the applications to control theory can be exploited much farther than in those references, so that it seems worthwhile to give a complete exposition of this subject starting from the very beginning. This will be attempted in the present paper.

2. Notation.

Let $X = \{x\}$ denote a complete locally compact metric space; its points represent the "states" of a given system. The independent variable $t \in R$ will be called time. Point sets in X-space will be denoted by capital letters A, B, ...; collections of such sets by script capitals $\mathcal{A}, \mathcal{B}, \ldots$.

In order to avoid infinite distances between sets we may replace the given metric $d(a, b)$ by

$$(2.1) \qquad \rho(a, b) = \frac{d(a, b)}{1 + d(a, b)} .$$

Furthermore, we define the distance between points and sets, and between sets, by:

$$(2.2) \qquad \rho(a, B) = \rho(B, a) = \inf\{\rho(a, b); b \in B\}$$

$$(2.3) \qquad \rho^*(A, B) = \sup\{\rho(a, B); a \in A\}.$$

$$(2.4) \qquad \rho(A, B) = \rho(B, A) = \max(\rho^*(A, B), \rho^*(B, A)).$$

Accordingly we define an ϵ-neighboring set of a given set A_o as

$$S_\epsilon(A_o) = \{x \in X; \rho(x, A_o) < \epsilon\}.$$

Notations which also will be used in this paper are the following:

$\mathcal{A}(X)$ = the collection of all non-empty subsets of X.
$\mathcal{B}(X)$ = the collection of all non-empty bounded subsets of X.
$\mathcal{K}(X)$ = the collection of all non-empty closed subsets of X.
$\mathcal{C}(X)$ = the collection of all non-empty compact subsets of X.

It is well known that (2.4) defines a pseudo-metric in the set of all non-void subsets of X ($\rho(A, B) = 0$ if and only if $\bar{A} = \bar{B}$, $\bar{A}$

denoting the closure of A). Restricting the attention to closed subsets, (2.4) defines a true metric (called the Hausdorff metric). For further details about the topology of spaces of subsets of X, the reader is referred to [6], [7].

3. The attainability function.

A control system will be given by a function which we call "attainability function" and has the following intuitive meaning. The evolution of a general control system is determined by its initial state x, the time t and some control action. Therefore, a given starting point x_o, t_o and a time $t_1 > t_o$ determine a whole set of possible end states $x(t_1)$, corresponding to all possible choices of the control action. This "attainable set from x_o, t_o at time t_1" will be denoted by $F(x_o, t_o, t_1)$.

The following axioms will be assumed:

I) $F(x_o, t_o, t)$ is a closed non-empty subset of X, defined for every $x \in X$; $t, t_o \in R$; $t_o \leq t$.

II) Initial condition: $F(x_o, t_o, t_o) = \{x_o\}$ for every x_o, t_o.

III) Semigroup property: for $t_o \leq t_1 \leq t_2$

$$F(x_o, t_o, t_2) = \bigcup_{x_1 \in F(x_o,t_o,t_1)} F(x_1, t_1, t_2).$$

IV) Given x_1, t_1, $t_o \leq t_1$, there exists an x_o such that

$$x_1 \in F(x_o, t_o, t_1).$$

V) $F(x_o, t_o, t)$ is continuous in t: given x_o, $t_o \leq t_1$, $\epsilon > 0$, there is a $\delta > 0$ such that $\rho(F(x_o, t_o, t), F(x_o, t_o, t_1)) < \epsilon$ for $|t - t_1| < \delta$.

VI) $F(x_o, t_o, t)$ is upper semicontinuous in (x_o, t_o), uniformly in any finite interval $t \in T = [t_1, t_2]$, $t_o \leq t_1 \leq t_2$: given x_o, t_o, t_1, t_2 and $\epsilon > 0$, there is a $\delta > 0$ such that

$$\rho^*(F(x_o', t_o', t), F(x_o, t_o, t)) < \epsilon$$

for all x_o', t_o', t satisfying

$$\rho(x_o', x_o) < \delta, \quad |t_o - t_o'| < \delta, \quad t_1 \leq t \leq t_2.$$

It is interesting to note that axiom III is equivalent to the following two together:

III a) If $x_1 \in F(x_o, t_o, t_1)$ and $x_2 \in F(x_1, t_1, t_2)$, then $x_2 \in F(x_o, t_o, t_2)$.

III b) If $x_2 \in F(x_o, t_o, t_2)$ and $t_o \leq t_1 \leq t_2$, there exists an $x_1 \in F(x_o, t_o, t_1)$ such that $x_2 \in F(x_1, t_1, t_2)$.

Each of the axioms (III a) and III b) are independent of the remaining set, as the following example shows.

<u>Example 3.1</u>. $X = R^1$. $x \in F(x_o, t_o, t)$ if and only

$$\text{if} \begin{cases} |x - x_o| \leq \alpha(t - t_o) & \text{for } t - t_o \leq T > 0, \\ |x - x_o| \leq \alpha T + \beta(t - t_o - T) & \text{for } t - t_o \geq T. \end{cases}$$

If $0 < \alpha < \beta$ (figure 1), the system satisfies axioms 1, 2, 3a, 4 and 5 but not 3b.

If $0 < \beta < \alpha$ (figure 2), the system satisfies axioms 1, 2, 3b, 4 and 5 but not 3a.

The independence of axiom IV is proved by the next example.

<u>Example 3.2</u>. $X = R^1$.

$$F(x_o, t_o, t) = \{x\} = \{\frac{x_o}{1 + |x_o|(t-t_o)}\}. \quad (t \geq t_o).$$

The relation between x_o and x is

$$\frac{1}{x} = \frac{1}{x_o} + t - t_o \quad \text{for} \quad x_o, \ x > 0$$

$$\frac{1}{x} = \frac{1}{x_o} - (t - t_o) \text{ for } \quad x_o, \ x < 0.$$

Therefore

$$x_o = \frac{x}{1 - |x| \, (t - t_o)}$$

which tends to infinity for $t - t_o \to \frac{1}{|x|}$, and for $t_o \leq t - \frac{1}{|x|}$ there is no point x_o satisfying the axiom IV, in spite of the fact that all the other axioms are obviously fulfilled (figure 3).

4. Properties of the attainability function.

Lemma 4.1: The attainability function $F(x, t_o, t)$ is upper semicontinuous in the triple (x, t_o, t).

This follows directly from the inequality

$$\rho^*(F(x_o', t_o', t'), F(x_o, t_o, t)) \leq$$

$$\leq \rho^*(F(x_o', t_o', t'), F(x_o, t_o, t')) +$$

$$+ \rho(F(x_o, t_o, t'), F(x_o, t_o, t)).$$

Indeed, given x_o, t_o, t and $\epsilon > 0$, one can find $\delta_1 > 0$ such that for $|t' - t| < \delta_1$ the last term is less than $\frac{\epsilon}{2}$ and then find δ_2 such that the first term of the right member is less than $\frac{\epsilon}{2}$ for $\rho(x_o', x_o) < \delta_2$, $|t_o' - t_o| < \delta_2$, uniformly in the interval $|t' - t| \leq \delta_1$.

Lemma 4.2: If X and Y are complete locally compact metric spaces, and if the function F: $X \to \mathcal{Q}(Y)$ is upper semicontinuous in the sense that given $x_o \in X$, $\epsilon > \rho$, there is a $\delta > 0$ such that $\rho(x, x_o) < \delta$ implies $\rho^*(F(x), F(x_o)) < \epsilon$, then the function $F: \mathcal{Q}(X) \to \mathcal{Q}(Y)$ defined by

$$F(A) = \bigcup_{x \in A} F(x) \qquad (A \subset X; A \neq \emptyset)$$

is upper semicontinuous for compact sets; more exactly, if $A_o \subset X$ is compact (and non-empty), then for any $\epsilon > 0$ there is a $\delta > 0$ such that

$\rho^*(A, A_o) < \delta$ implies $\rho^*(F(a), F(a_o)) < \epsilon$.

Proof: The conditions for upper semi-continuity can be written

$$F(S_\delta(x_o)) \subset S_\epsilon(F(x_o))$$

respectively

$$F(S_\delta(A_o)) \subset S_\epsilon(F(A_o)).$$

Each point $x \in A_o$ has a neighborhood $V(x)$ such that $F(V(x)) \subset S_\epsilon(F(x)) \subset S_\epsilon(F(A_o))$. The set

$$V_o = \bigcup_{x \in A_o} V(x)$$

is open and contains the compact set A_o, therefore there is a $\delta > 0$ such that $S_\delta(A_o) \subset V_o$. Hence

$$F(S_\delta(A_o)) \subset F(V_1) = \bigcup_{x \in A_o} F(V(x)) \subset S_\epsilon(F(A_o)).$$

<u>Theorem 4.1</u>: If $A_o \subset X$, $T_o \subset R$, $T_1 \subset R$ are compact non-empty sets and for any pair $t_o \in T_o$, $t_1 \in T_1$ the relation $t_o \leq t_1$ holds, then the attainability function

$$F(A_o, T_o, T_1) = \bigcup_{\substack{x \in A_o \\ t_o \in T_o \\ t \in T_1}} F(x, t_o, t)$$

is upper semi-continuous at (A_o, T_o, T_1); more exactly, given A_o, T_o, T_1 and $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho^*(A, A_o) < \delta, \ \rho^*(T_o', T_o) < \delta, \ \rho^*(T_1', T_1) < \delta$$

imply

$$\rho^*(F(A, T_o', T_1'), F(A_o, T_o, T_1)) < \epsilon.$$

This is an immediate consequence of Lemma 4.2. It obviously can be formulated in a little more general way, taking in the (t_o, t_1)-plane, instead of the rectangle $T_o \times T_1$, any domain D such that for any pair $(t_o, t_1) \epsilon D$ the relation $t_o \leq t_1$ holds.

Lemma 4.3. If $A \subset X$ is compact, there is an $\epsilon > 0$ such that the closure of $S_\epsilon(A)$ is compact.

Proof: X being locally compact, each $x \epsilon A$ has a neighborhood $V(x)$ with compact closure. From $\bigcup_{x \epsilon A} V(x)$ it is possible to select a finite covering of $A \subset \bigcup_{i=1}^{n} V(x_i)$. Then $V_o = \bigcup_{i=1} V(x_i)$ is an open set with compact closure. There is an ϵ-neighborhood of A contained in V_o and $\overline{S_\epsilon(A)}$, being a closed subset of the compact set $\overline{V_o}$, is also compact.

Lemma 4.4: If $A \subset X$ is not totally bounded, then there is an $\epsilon > 0$ such that if $\rho^*(A, B) < \epsilon$, then B is also not totally bounded.

(A set is called totally bounded if for any $\epsilon > 0$ it can be covered by a finite number of sets of diameter less than ϵ).

Proof: Assuming $\rho^*(A, B) < \epsilon$ and B totally bounded, it is possible to cover B with a finite number of balls of diameter 2ϵ:

$$B \subset \bigcup_{i=1}^{n} S_\epsilon(x_i).$$

Each point of A is at distance less than ϵ from some ball $S_\epsilon(x_i)$ and therefore at distance less than 2ϵ from x_i. In consequence

$$A \subset \bigcup_{i=1}^{n} S_{2\epsilon}(x_i)$$

and this can be done with any $\epsilon > 0$. Therefore A is totally bounded, contrary to the hypothesis.

<u>Theorem 4.2</u>: If $x \in X$, $t_1 \geq 0$, and F is the attainability function of a generalized control system, then $F(x, t_o, t_1)$ is compact.

Proof: $F(x, t_o, t_1)$ is closed by axiom I, so that it has to be proved that it is totally bounded. The set of values of $t \in [t_o, t_1]$ for which $F(x, t_o, t)$ is not totally bounded (assuming it is non-empty) has a greatest lower bound t^*. In any neighborhood of t^*, there are values t', t'' such that $F(x, t_o, t')$ is totally bounded and $F(x, t_o, t'')$ is not. By continuity of $F(x, t_o, t)$ with respect to t and for any given $\epsilon > 0$, the neighborhood of t^* can be chosen so that for any value t of it, $\rho(F(x, t_o, t^*), F(x, t_o, t)) < \epsilon$. Assuming

$F(x, t_o, t^*)$ totally bounded and therefore compact, this contradicts Lemma 4.3. Assuming $F(x, t_o, t^*)$ not totally bounded, it contradicts Lemma 4.4. Therefore t^* does not exist and as $F(x, t_o, t)$ is compact for $t = t_o$, it is also compact for all $t > t_o$.

<u>Theorem 4.3:</u> If $A \subset X$ is compact and $t \geq t_o$, then $F(A, t_o, t)$ is compact.

Proof: The set $F(x, t_o, t)$ is compact for each $x \in A$ and by Lemma 4.3 there is a neighborhood $W(F(x, t_o, t))$ with compact closure. By the upper semi-continuity of the attainability function (axiom VI) there is a neighborhood $V(x)$ such that $F(V(x), t_o, t) \subset W(F(x, t_o, t))$. As A is compact there is a finite set $x_i \in A$, $(i = 1,2, \ldots, n)$ such that $V_o = \bigcup_{i=1}^{n} V(x_i) \supset A$. Therefore

$$F(V_o, t_o, t) = \bigcup_{i=1}^{n} F(V(x_i), t_o, t) \supset F(A, t_o, t)$$

and the set $F(A, t_o, t)$ is contained in the compact set

$$\bigcup_{i=1}^{n} \overline{W(F(x_i, t_o, t))}$$

and is totally bounded. The proof will be complete by showing that it is also closed.

If $y_i \in F(A, t_o, t)$, $(i = 1,2,3, \ldots)$, and $\lim_{i \to \infty} y_i = y_o$, there are points $x_i \in A$ $(i = 1,2,3, \ldots)$ such that $y_i \in F(x_i, t_o, t)$.

By compactness of A, some subsequence x_i converges to some $x_o \in A$; from the upper semi-continuity and closedness of $F(x, t_o, t)$, the result $y_o \in F(x_o, t_o, t)$ is easily obtained.

<u>Corollary 4.1</u>: If A is totally bounded, so is $F(A, t_o, t)$. Indeed, $\bar{A}$ is compact and $F(A, t_o, t)$ is a subset of the compact set $F(\bar{A}, t_o, t)$.

<u>Corollary 4.2</u>: If $A \subset X$ is compact, $T_o \subset R$ is compact and $T \subset R$ is compact and such that $t_o \leq t$ for any pair $t_o \in T_o$, $t \in T$, then $F(A, T_o, T)$ is also compact.

The proof is easily obtained replacing "$x \in X$" by "$(x, t_o, t) \in X \times R^2$" in the proof of Theorem 4.3. It can even be generalized to the case $(t_o, t) \in V \subset R^2$ such that if $(t_o, t) \in V$, then $t_o \leq t$.

<u>Theorem 4.4</u>: If $A \subset X$ and $T_o \subset R$ are compact sets, then the attainability function $F(A, T_o, \tau)$ is continuous in τ, where it is defined (i.e. for $\tau \geq$ all $t_o \in T_o$).

Proof: From Theorem 4.1 it follows that, given $\tau_o \geq$ all $t_o \in T_o$ and given $\epsilon > 0$, there exists $\delta > 0$ such that for $|\tau - \tau_o| < \delta$ and $\tau \geq$ all $t_o \in T_o$,

$$\rho^*(F(A, T_o, \tau), \ F(A, T_o, \tau_o)) < \epsilon,$$

so it remains only to prove the opposite relation, $\rho^*(F(A, T_o, \tau_o), F(A, T_o, \tau)) < \epsilon$.

For any point $x_o \in F(A, T_o, \tau_o)$ there is some $\delta(x_o)$ such that for $|\tau - \tau_o| < \delta(x_o)$ and $\tau \geq$ all $t_o \in T_o$,

$$\rho(x_o, F(A, T_o, \tau)) < \epsilon.$$

Indeed, to prove this it suffices to take $x \in F(A, T_o, \tau)$ in the following way:

i) if $\tau > \tau_o$, take $x \in F(x_o, \tau_o, \tau) \subset F(A, T_o, \tau)$;

ii) if $\tau < \tau_o$, take $x \in F(A, T_o, \tau)$ and such that $x_o \in F(x, \tau, \tau_o)$, which is possible by axiom III b;

iii) if $\tau = \tau_0$, take $x = x_o$.

In all three cases, $\tau \to \tau_o$ implies $x \to x_o$ which proves that $\rho(x, x_o) \to 0$ and

$$\lim_{\tau \to \tau_o} \rho(F(A, T_o, \tau), x_o) = 0.$$

Now suppose that $\lim_{\tau \to \tau_o} \rho^*(F(A, T_o, \tau_o), F(A, T_o, \tau)) = 0$ is false. Then there are sequences $\tau_i \to \tau_o$ and $x_i \in F(A, T_o, \tau_o)$, $(i = 1,2,3, \ldots)$ such that $\rho(x_i, F(A, T_o, \tau_i) > a > 0$. By compactness, the sequence x_i has a limit point and some subsequence $x_i' \to x_o \in F(A, T_o, \tau_o)$. But then there is some n such that for $i' > n$, $\rho(x_i', x_o) < \frac{a}{2}$ and $\rho(x_o, F(A, T_o, \tau_i') < \frac{a}{2}$ in contradiction with $\rho(x_i', F(A, T\ , \tau_i')) > a$.

Lemma 4.5: If the function $F: [t_o, t_1] \to \mathcal{C}(X)$ is continuous and $F(t_o)$ is connected, then $F([t_o, t_1])$ is a continuum (compact and connected).

Proof: To show that it is compact, consider a sequence $x_i \in F([t_o, t_1]$ $(i = 1,2,3, \ldots)$. Obviously $x_i \in F(\tau_i)$, $\tau_i \in [t_o, t_1]$ and in order to prove that the sequence x_i has a limit point one may assume $\tau_i \to \tau_o$ for $i \to \infty$. Now

$$\rho(x_i, F(\tau_o)) \leq \rho(x_i, F(\tau_i)) + \rho(F(\tau_i), F(\tau_o));$$

the first term of which is zero and the second tends to zero by continuity. Hence

$$\lim_{i \to \infty} \rho(x_i, F(\tau_i)) = 0.$$

As $F(\tau_o) \in \mathcal{C}(X)$ is compact, almost every x_i belongs to some compact neighborhood $\overline{S_\epsilon(F(\tau_o))}$, which proves the existence of a limit point $x_o \in F(\tau_o) \subset F([t_o, t_1])$.

In order to prove that it is connected, assume that $F([t_o, t_1] = F_1 \cup F_2$ is a separation (i.e. F_1 and F_2 are closed and disjoint). Consider $F([t_o, t])$ as a function of t: for $t = t_o$ it is connected, for every $t \in [t_o, t_1]$ it is compact and it is continuous in t (which follows easily from the continuity of $F(t)$). Moreover, it is nondecreasing: if $t_o \leq t' \leq t''$ then $F(t') \subset F(t'')$. Therefore

$F(t_o) \subset F_1 \cup F_2$ and as $F(t_o)$ is connected one may suppose that $F(t_o) \subset F_1$.

Divide the interval $[t_o, t_1]$ in two sets: those values of t for which $F([t_o, t]) \subset F_1$ and those values for which $F([t_o,t]) \cap F_2 \neq \emptyset$. It is easy to see that because both sets are closed and supposedly non-empty, they define a separation of the interval $[t_o, t_1]$, which is absurd because $[t_o, t_1]$ is connected.

An immediate consequence of this lemma is the following theorem.

<u>Theorem 4.5.</u> If $A \subset X$ is a continuum, $t_o \leq t_1$ and $F(x, t_o, t)$ the attainability function, then $F(A, t_o, [t_o, t_1])$ is a continuum.

5. <u>The attainability function for $t < t_o$.</u>

The domain of definition of the attainability function $F(x, t_o, t)$ can be extended in a natural way for the values $t < t_o$. Almost all basic properties are maintained; the only exception being the continuity condition which is not satisfied in the strong form of axiom V.

In order to distinguish between the function already defined and the extension to be defined now, the notation $G(x, t_o, t)$ will be used for the extension.

<u>Definition:</u> The function $G(x, t_o, t)$; $\{x \in X; t_o \in R; t \in R; t \leq t_o\} \to \mathcal{Q}(X)$ is defined by

$$y \in G(x, t_o, t) \Longleftrightarrow x \in F(y, t, t_o).$$

Note that the relation between F and G is reciprocal, but they are not "inverse" functions of each other.

Proposition 5.1: If $x \in X$ and $t_o \geq t$, then $G(x, t_o, t)$ is a closed non-empty subset of X.

Proof: Suppose $y_i \in G(x, t_o, t)$ $(i = 1,2,3, \ldots)$ and $\lim_{i \to \infty} y_i = y$. Then $x \in F(y_i, t, t_o)$ and

$$\rho(x,F(y,t,t_o)) \leq \rho(x,F(y_i,t,t_o)) + \rho^*(F(y_i,t,t_o), F(y,t,t_o)).$$

Now, $\rho(x, F(y_i, t, t_o)) = 0$ and $\lim_{i \to \infty} \rho^*(F(y_i,t,t_o), F(y,t,t_o)) = 0$, therefore $\rho(x, F(y, t, t_o)) = 0$. As $F(y, t, t_o)$ is closed, $x \in F(y, t, t_o)$ and $y \in G(x, t_o, t)$. This proves that $G(x, t_o, t)$ is closed. That it is non-empty follows from axiom IV.

Proposition 5.2: $G(x, t_o, t_o) = \{x\}$.

Proposition 5.3: If $x_o \in X$ and $t_o \geq t_1 \geq t_2$, then

$$G(x_o, t_o, t_2) = \bigcup_{x_1 \in G(x_o,t_o,t_1)} G(x_1, t_1, t_2).$$

Proposition 5.4: Given $x \in X$, $t_o \geq t_1$, there exists a $y \in X$ such that $x \in G(y, t_o, t_1)$.

The proofs of these propositions are straightforward.

The function $G(x, t_o, t)$ is not, in general, continuous in t. A counterexample follows, in which the set $G(x, t_o, t)$ becomes unbounded.

Example 5.1:

$$F(x, t_o, t) = \begin{cases} [\frac{x}{1 + x(t - t_o)}, x] \ldots & \text{for } x > 0 \\ x \ldots\ldots\ldots\ldots & \text{for } x \leq 0. \end{cases}$$

Then (see fig. 4)

$$G(y, t, t_o) = \begin{cases} [y, \frac{y}{1 - y(t - t_o)}] \ldots & \text{for } y > 0,\ y(t-t_o) < 1 \\ [y, \infty) \ldots\ldots\ldots\ldots & \text{for } y > 0,\ y(t-t_o) \geq 1 \\ y \ldots\ldots\ldots\ldots & \text{for } y \leq 0. \end{cases}$$

Lemma 5.1: If $y_o \in F(x_o, t_o, t_1)$, $t_1 \geq t_o$, then the set

$$A_o = F(x_o, t_o, [t_o, t_1]) \cap G(y_o, t_1, [t_o, t_1])$$

is a continuum, i.e. compact and connected.

Proof: Consider the set

$$A(\tau) = F(x_o, t_o, \tau) \cap G(y_o, t_1, \tau)$$

for $t_o \leq \tau \leq t_1$. It is non-empty by virtue of axiom III b. It is compact because it is the intersection of a compact and a closed set. It will be proved that it is continuous in τ, i.e.

$$\rho(A(\tau), A(\tau_o)) = \max[\rho^*(A(\tau), A(\tau_o)), \rho^*(A(\tau_o), A(\tau))] \to 0$$

for $\tau \to \tau_o$.

If, for $i = 1,2,3, \ldots, z_i \to z_o, \ \tau_i \to \tau_o, \tau_o, \tau_i \in [t_o, t_1]$ and $z_i \in A(\tau_i)$, then by continuity of the attainability function

$$z_o \in F(x_o, t_o, \tau_o).$$

Besides,

$$z_i \in G(y_o, t_1, \tau_i)$$

is equivalent to

$$y_o \in F(z_i, \tau_i, t_1)$$

and by semi-continuity and closedness of F,

$$y_o \in F(z_o, \tau_o, t_1)$$

or

$$z_o \in G(y_o, t_1, \tau_o),$$

so that

$$z_o \in A(\tau_o).$$

This proves that

$$\lim_{\tau \to \tau_o} \rho^*(A(\tau_o), \quad A(\tau)) = 0$$

it is sufficient to show that given $z_o \in A(\tau_o)$ and $\tau_i \to \tau_o$ ($i = 1,2,3, \ldots, \tau_i > t_o$) there exist $z_i \in A(\tau_i)$ such that $z_i \to z_o$. The cases $\tau_i \geq \tau_o$ and $\tau_i < \tau_o$ will be treated separately and the general case follows as a combination of both.

Suppose $\tau_i > \tau_o$. The set

$$F(z_o, \tau_o, \tau_i) \cap G(y_o, t_1, \tau_i)$$

is a subset of $A(\tau_i)$; it is non-empty because $y_o \in F(z_o, \tau_o, t_1)$. Taking as z_i any point of this set, it follows that

$$\lim_{i \to \infty} z_i \in \lim_{i \to \infty} F(z_o, \tau_o, \tau_i) = \{x_o\}.$$

Suppose now $\tau_i < \tau_o$ and take

$$z_i \in G(z_o, \tau_o, \tau_i) \cap F(x_o, t_o, \tau_i),$$

which is a non-empty subset of $A(\tau_i)$. As $z_i \in F(x_o, t_o, [t_o, t_1])$ and this set is compact, the sequence z_i has some limit point and it may be assumed $z_i \to \zeta$. It will be proved that $\zeta = z_o$. Indeed,

$$z_o \in F(z_i, \tau_i, \tau_o)$$

and from the semi-continuity of the attainability function,

$$z_o \in F(\zeta, \tau_o, \tau_o) = \{\zeta\}$$

follows.

Having proved that $A(\tau)$ is continuous, the desired result follows from Lemma 4.5.

<u>Theorem 5.1</u>: If $G(x_o, t_o, \tau_o)$ is compact and $t_o \geq \tau_o$, the function $G(x, t, \tau)$ is upper semi-continuous at (x_o, t_o, τ_o), i.e. given $\epsilon > 0$ there is a $\delta > 0$ such that for $\rho(x, x_o) < \delta$, $|t - t_o| < \delta$, $|\tau - \tau_o| < \delta$

$$\rho^*(G(x, t, \tau), \ G(x_o, t_o, \tau_o)) < \epsilon.$$

Proof. If the theorem were false, it would be possible to determine sequences

$$x_i \to x_o, \ t_i \to t_o, \ \tau_i \to \tau_o, \ t_i \geq \tau_i, \ y_i \in G(x_i, t_i, z_i), \quad (i = 1,2,3, \ldots)$$

such that the sequence y_i has no limit point belonging to $G(x_o,t_o,\tau_o)$; it will be proved that this assumption leads to a contradiction.

Consider first the case when the sequence y_i has some limit point y_o. Taking a subsequence one may write $y_i \to y_o$ and it will be proved that $y_o \in G(x_o, t_o, \tau_o)$. Indeed,

$$y_i \in G(x_i, t_i, \tau_i)$$

implies

$$x_i \in F(y_i, \tau_i, t_i)$$

and by the upper semi-continuity and closedness of $F(y_o, \tau_o, t_o)$,

$$x_o \in F(y_o, \tau_o, t_o).$$

Now the case of y_i not having any limit point has to be ruled out. The set $G_o = G(x_o, t_o, \tau_o)$ is compact by hypothesis, so that the set

$$H = F(G_o, \tau_o, [\tau_o, t_o])$$

is also compact and $x_o \in H$, (fig. 5). Therefore there is some sphere $\Sigma(a, r)$ of center a and radius r containing H in its interior. As $x_i \to x_o$ it may be assumed that x_i is also interior to $\Sigma(a, r)$.

Suppose the sequence y_i has no limit point. Then, disregarding a finite number of terms, y_i is exterior to $\Sigma(A, r)$. Now, the set

$$F(y_i, \tau_i, [\tau_i, t_i]) \cap G(x_i, t_i, [\tau_i, t_i])$$

is a continuum joining y_i and x_i (Theorem 4.4); therefore it meets $\Sigma(a, r)$ in some point z_i and

$$z_i \in G(x_i, t_i, \tau_i')$$

for some $\tau_i' \in [\tau_i, t_i]$. But z_i belongs to the compact set $\Sigma(a, r)$, so that for some subsequence $z_i \to z_o$ and $\tau_i' \to \tau_o' \in [\tau_o, t_o]$. Now

$$x_i \in F(z_i, \tau_i', t_i)$$

implies, as before

$$x_o \in F(z_o, \tau_o', t_o)$$

and

$$z_o \in G(x_o, t_o, \tau_o') = G_o'.$$

Applying the elementary properties of the function G, one sees that

$$G_o' \subset F(G(G_o', \tau_o', \tau_o), \tau_o, [\tau_o, t_o]) =$$

$$= F(G(x_o, t_o, \tau_o), \tau_o, [\tau_o, t_o]) = H.$$

This last set is contained in the interior of $\Sigma(a, r)$ and so must be z_o, but on the other hand $z_i \in \Sigma(a, r)$ implies that z_o lies on the (boundary of that) sphere, which gives the desired contradiction.

<u>Remark</u>: If $\tau_o \le t_o$ and $A \subset X$ and $G(A, t_o, \tau_o)$ are compact, then $G(A, t_o, \tau)$ is compact for all $\tau_o \le \tau \le t_o$. Indeed

$$G(A, t_o, \tau) \subset F(G(A, t_o, \tau_o), \tau_o, \tau),$$

which is compact. $G(A, t_o, \tau)$ is therefore totally bounded, and as it is closed, it is compact.

<u>Corollary 5.1</u>: $G(x_o, t_o, \tau_o)$ is continuous at $t_o = \tau_o$. According to the preceding theorem

$$\rho(G(x, t, \tau), G(x_o, t_o, t_o)) = \rho(G(x, t, \tau), x_o) =$$

$$= \rho^*(G(x, t, \tau), G(x_o, t_o, t_o)) \to 0$$

for $x \to x_o$, $t \to t_o$, $\tau \to t_o$, $\tau \leq t$.

<u>Theorem 5.2</u>: If $G(x_o, t_o, \tau_o)$ is compact, $t_o \geq \tau_o$, then the function $G(x_o, t_o, \tau)$ is continuous in τ at $\tau = \tau_o$.

Proof: If $\tau_o = t_o$ this result is included in the preceding corollary, so that $\tau_o < t_o$ may be assumed. For $\tau \to \tau_o$,

$$\rho^*(G(x_o, t_o, \tau), G(x_o, t_o, \tau_o)) \to 0$$

is a consequence of Theorem 5.1, and only

$$\rho^*(G(x_o, t_o, \tau_o), G(x_o, t_o, \tau)) \to 0$$

remains to be proved. As $G(x_o, t_o, \tau_o)$ is compact, this is equivalent to the condition: given any $y_o \in G(x_o, t_o, \tau_o)$ and any sequence

$\tau_i \to \tau_o$, $(\tau_i < t_o$, $i = 1,2,3, \ldots,$ there is a sequence $y_i \in G(x_o,t_o,\tau_i))$ such that $y_i \to y_o$ for $i \to \infty$. This sequence y_i is easily constructed in the following way:

i) If $\tau_i < \tau_o$, take $y_i \in G(y_o, \tau_o, \tau_i) \subset G(x_o, t_o, \tau_i)$.

ii) If $\tau_i > \tau_o$, take $y_i \in G(x_o, t_o, \tau_i) \cap F(y_o, \tau_o, \tau_i)$.

iii) If $\tau_i = \tau_o$, take $y_i = y_o \in G(x_o, t_o, \tau_o)$.

In all three cases, for $i \to \infty$, $\tau_i \to \tau_o$ and by continuity of $F(x, t, \tau)$ or $G(x, t, \tau)$ (at $t = \tau$ in this last case), the result $y_i \to y_o$ is obtained (see fig. 6).

Theorem 5.3: If $A_o \subset X$ and $G(A_o, t_o, \tau_o)$ are compact, $\tau_o \leq t_o$ and $\epsilon > 0$, then there is $\delta > 0$ such that

$$\rho^*(G(A, t, \tau), G(A_o, t_o, \tau)) < \epsilon$$

for all $A \subset X$, $t \geq \tau$, such that $\rho^*(A, A_o) < \delta$, $|t - t_o| < \delta$, $\tau \in [\tau_o, t_o]$.

Briefly speaking, as long as $G(A, t, \tau)$ is compact, it is upper semi-continuous in (A, t) uniformly in any finite τ-interval.

Proof: Assuming the theorem to be false, there are some sequences $A_i \subset X$, $t_i \to t_o$, $z_i \in [\tau_o, t_o]$, $\tau_i \leq t_i$ such that $\rho^*(A_i, A_o) \to 0$ and

$$\rho^*(G(A_i, t_i, \tau_i), G(A_o, t_o, \tau_i)) > a > 0 \quad (i = 1,2,3, \ldots).$$

This means that there are points $x_i \in A_i$ $(i = 1,2,3, \ldots)$ such that

$$\rho^*(G(x_i, t_i, \tau_i), G(A_o, t_o, \tau_i)) > a.$$

As $\rho^*(A_i, A_o) \to 0$, one may assume that all A_i $(i = 1,2,3, \ldots)$ are contained in some compact set B. Therefore some subsequence $x_i \to x_o$, where $x_o \in A_o$. Taking subsequences, one may write also $\tau_i \to \tau' \in [\tau_o, t_o]$.

By Theorem 5.1, therefore

$$\lim_{i \to \infty} \rho^*(G(x_i, t_i, \tau_i), G(x_o, t_o, \tau')) = 0.$$

But as $x_o \in A_o$, also

$$\lim_{i \to \infty} \rho^*(G(x_i, t_i, \tau_i), G(A_o, t_o, \tau')) = 0.$$

By Theorem 5.2,

$$\lim_{i \to \infty} \rho^*(G(x_i, t_i, \tau_i), G(A_o, t_o, \tau_i)) = 0$$

in contradiction with the assumption.

Comparison of the F and G-functions. Propositions 5.1, 5.2, 5.3 and 5.4 and Theorems 5.2 and 5.3 show that, as long as $G(x, t_o, t)$ remains totally bounded, it satisfies all the axioms of the function $F(x,t_o,t)$, but $G(x, t_o, t)$ may become totally unbounded (example 5.1), thus violating the continuity in t.

Even disregarding this difference, it is confusing to treat F and G as the same function, because the axiom III (semi-group property), satisfied separately by F and G, give rise to the following weaker relations when combining both functions:

$$\left.\begin{aligned} G(F(x, t_o, t_2), t_2, t_1) &\supset F(x, t_o, t_1) \\ G(F(x, t_1, t_2), t_2, t_o) &\supset G(x, t_1, t_o) \\ F(G(x, t_2, t_o), t_o, t_1) &\supset G(X, t_2, t_1) \\ F(G(x, t_1, t_o), t_o, t_2) &\supset F(x, t_1, t_2) \end{aligned}\right. \qquad t_o \leq t_1 \leq t_2$$

where the inclusion sign cannot, in general, be replaced by the equality sign. The proof of these relations is obvious (fig. 7).

6. <u>Trajectories</u>.

<u>Lemma 6.1</u>: Let $F(x, t, \tau)$ and $G(x, \tau, t)$ be the attainability functions of a generalized control system, and φ: $[t_o, t_1] \to X$ a not necessarily continuous mapping such that $t_o \leq t_a \leq t_b \leq t_1$ imply $\varphi(t_b) \in F(\varphi(t_a), t_a, t_b)$. Then $x = \varphi(t)$ is continuous.

Proof: Suppose t_a fixed and $t \to t_a$, $(t, t_a \in [t_o, t_1])$.
Then:

i) if $t > t_a$, $\varphi(t) \in F(\varphi(t_a), t_a, t)$ and $\varphi(t) \to \varphi(t_a)$ by axiom V;

ii) if $t < t_a$, $\varphi(t) \in G(\varphi(t_a), t_a, t)$ and $\varphi(t) \to \varphi(t_a)$ by Theorem 5.2.

Definitions. A trajectory of a generalized control system is a mapping φ: $[t_o, t_1] \to X$ such that $t_o \leq t_a \leq t_b \leq t_1 \implies \varphi(t_b) \in F(\varphi(t_a), t_a, t_b)$.

A trajectory φ_1: $[t_a, t_b]$ is a prolongation of the trajectory φ_2: $[t_c, t_d]$ if $[t_c, t_d] \subset [t_a, t_b]$ and $\varphi_1(t) = \varphi_2(t)$ on $[t_c, t_d]$.

Sometimes it is convenient to consider a trajectory in the state-time space ψ: $[t_o, t_1] \to X \times R$, defining $\psi(t) = (\varphi(t), t)$. In the state-time space a trajectory is a Jordan arc. Indeed, it is continuous and without multiple points.

As usual in dynamical systems positive (and negative) half trajectories starting at some (x_o, t_o) will be considered sometimes.

Theorem 6.1: If, for a certain generalized control system, $x_1 \in F(x_o, t_o, t_1)$, then there exists a trajectory $\varphi(t)$ defined in $[t_o, t_1]$ such that $\varphi(t_o) = x_o$, $\varphi(t_1) = x_1$.

Proof: Assuming, for simplicity, $t_o = 0$, $t_1 = 1$, a trajectory satisfying the desired boundary conditions can be constructed in the following way:

For $t = 1/2, 1/4, 3/4, 1/8, 3/8, \ldots, \frac{p}{2^q}, \ldots$ the values of $\varphi(t)$ can be chosen successively such that

$$\varphi(\frac{p}{2^q}) \in F(\varphi(\frac{p-1}{2^q}), \frac{p-1}{2^q}, \frac{p}{2^q}) \cap G(\varphi(\frac{p+1}{2^q}), \frac{p+1}{2^q}, \frac{p}{2^q}),$$

defining $\varphi(t)$ for all binary fractions and obviously satisfying the definition of a trajectory. For the remaining values of t,

$$\bigcap_{\substack{t',\, t'' \text{ binary fractions} \\ t' < t < t''}} [F(\varphi(t'),t',t) \cap G(\varphi(t''),t'',t)] = K(t)$$

is not void because $K(t)$ is the intersection of compact sets with the finite intersection property. It is easy to see that $K(t)$ is a single point, but taking anyway $\varphi(t) \in K(t)$, this satisfies the relation defining a trajectory. Indeed, if for example $t_a < t_c \in [0, 1]$ are not binary fractions, there is a binary fraction t_b, such that $t_a < t_b < t_c$. Therefore

$$\varphi(t_a) \in K(t_a) \subset G(\varphi(t_b), t_b, t_a),$$

$$\varphi(t_c) \in K(t_c) \subset F(\varphi(t_b), t_b, t_c),$$

from which it follows that

$$\varphi(t_c) \in F(\varphi(t_a), t_a, t_c).$$

<u>Theorem 6.2</u> (Barbashin): If $\varphi_i(t)$, $(i = 1,2,3, \ldots)$ are trajectories of a certain generalized control system, which are defined in the interval $T_0 \le t \le T_1$, and if $\varphi_i(T_0) = x_i \to x_0$ for $i \to \infty$, then there is some subsequence $\varphi_{i_j}(t)$ converging to a trajectory $\varphi_0(t)$:

$$\lim_{j \to \infty} \varphi_{i_j}(t) = \varphi_o(t) \quad T_o \leq t \leq T_1,$$

and the convergence is uniform in the interval $[T_o, T_1]$.

Proof: As $x_i \to x_o$ it may be assumed that all $x_i \in S \subset X$ where S is compact, and hence $\varphi_i(t) \in S_t$ where $S_t = F(S, T_o, t)$ is also compact for any $t \in [T_o, T_1]$.

Taking any countable dense subset $\{t_i\}$ of the interval $[T_o, T_1]$, for example $t_o = T_o$, $t_1 = T_1$, $t_2 = \frac{T_o + T_1}{2}$, $t_n = T_o + \frac{p}{2^q}(T_1 - T_o)$, ... for any value t_n it is possible to choose a subsequence φ_i such that $\varphi_{i_{j,n}}(t_n)$ converges. Furthermore $\varphi_{i_{j,n}}(t_n)$ can be chosen to be a subsequence of the one corresponding to t_{n-1}. Then, according to the well known classical procedure, the diagonal sequence $\varphi_{i_{j,j}}$ converges on the whole dense set $\{t_n\}$, to values which will be denoted by $\varphi_o(t_n)$.

Consider two values $t_r < t_s$. It will be shown that $\varphi_o(t_s) \in F(\varphi_o(t_r), t_r, t_s)$. Take $\varphi_i(t)$ to be the diagonal sequence defined above which converges pointwise to $\varphi_o(t)$ on the dense subset $\{t_n\}$. Given any $\epsilon > 0$, there is n_1 such that $\rho(\varphi_i(t_s), \varphi_o(t_s)) \leq \frac{\epsilon}{2}$ for all $i \geq n_1$. A value $\delta > 0$ can also be determined such that $\rho^*(F(x, t_r, t_s), F(\varphi_o(t_r), t_r, t_s) \leq \frac{\epsilon}{2}$ for $\rho(x, \varphi_o(t_r)) \leq \delta$. Correspondingly, for some n_2, $\rho(\varphi_i(t_r), \varphi_o(t_r)) \leq \delta$ for $i \geq n_2$. Hence, for $i \geq \max(n_1, n_2)$:

$$\rho(\varphi_o(t_s), F(\varphi_o(t_r), t_r, t_s)) \leq \rho(\varphi_o(t_s), \varphi_i(t_s)) +$$

$$+ \rho(\varphi_i(t_s), F(\varphi_o(t_r), t_r, t_s) \leq$$

$$\rho(\varphi_o(t_s), \varphi_i(t_s)) + \rho^*(F(\varphi_i(t_r), t_r, t_s), F(\varphi_o(t_r), t_r, t_s)) \leq$$

$$\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore $\rho(\varphi_o(t_s), F(\varphi_o(t_r), t_r, t_s)) = 0$ and $\varphi_o(t_s) \in F(\varphi_o(t_r), t_r, t_s)$.

At the remaining values of t, $\varphi_o(t)$ can be defined by the same procedure used in Theorem 6.1, obtaining thus the trajectory $\varphi_o(t)$ defined in the whole interval $[T_o, T_1]$.

It remains to prove that $\varphi_i(t) \to \varphi_o(t)$ uniformly (until now the convergence has been proven only for the dense subset mentioned above). Assuming the contrary, there is some subsequence φ_{i_j} and corresponding values $t_j \in [T_o, T_1]$ $(j = 1,2,3, \ldots)$ such that $t_j \to t_o$ and $\varphi_{i_j}(t_j) \to y \neq \varphi_o(t_o)$ for $j \to \infty$.

If $t_o \leq T_1$, take any fixed value τ, $t_o \leq \tau \leq T_1$; if $t_o = T$, take $\tau = T_1$. Disregarding a finite number of terms, it may be assumed that $t_j < \tau$. Hence, $\varphi_o(\tau) \in F(\varphi_{i_j}(t_j), t_j, \tau)$, and by upper semi-continuity,

$$\varphi_o(\tau) \in F(y, t_o, \tau).$$

Therefore

$$\varphi_o(t_o) = \lim_{\tau \to t_o} \varphi_o(\tau) = \lim_{\tau \to t_o} F(y, t_o, \tau) = y,$$

which contradicts the assumption.

<u>Theorem 6.3</u>: If $\varphi_i(t)$, $(i = 1,2,3, \ldots)$ are trajectories of a certain generalized control system, which are defined in $T_o \leq t \leq T_1$, and if $\varphi_i(T_1) \to x_1$ for $i \to \infty$, and in addition $G(x_1, T_1, T_o)$ is compact, then there is some subsequence φ_{i_j} converging uniformly to a trajectory $\varphi_o(t)$.

The proof is the same as for the previous theorem.

<u>Theorem 6.4</u>: If $\varphi_i(t)$ are trajectories defined for $t \in [T_o, +\infty)$, and if $\varphi_i(T_o) \to x_o$ for $i \to \infty$, then there is a subsequence converging to some trajectory $\varphi_o(t)$ $(t \geq T_o)$, and the convergence is uniform in any finite time interval.

<u>Theorem 6.5</u>: If $\varphi_i(t)$ are trajectories defined for $t \in (-\infty, T_1]$, and if $\varphi_i(T_o) \to x_o$ for $i \to \infty$, and if $G(x_o, T_o, t)$ is compact for all $t \leq T_o$ then there is a subsequence converging to some trajectory $\varphi_o(t)$ $(t \leq T_o)$, and the convergence is uniform in any finite time interval.

<u>Theorem 6.6</u>: If $x_o \in X$ and for a certain general control system, $G(x_o, t_o, \tau)$ is compact for every τ, $t_1 < \tau \leq t_o$ but not for $\tau = t_1$, then there is a trajectory $\varphi(\tau)$ defined in $(t_1, t_o]$ passing through x_o, t_o which is unbounded for $\tau \to t_1$, ($\varphi(\tau)$ has no limit point for $\tau \to t_1$).

Indeed, it is sufficient to take a sequence $x_i \in G(x_o, t_o, t_1)$ without limit point (this is possible because $G(x_o, t_o, t_1)$ is not totally bounded) and a corresponding sequence of trajectories $\varphi_i(t)$ joining $\varphi_{i_j}(t)$ which converges for every $t \in (t_1, t_o]$ defining a limit trajectory $\varphi_o(t)$, $(t_1 < t \leq t_o)$, but obviously for $t = t_1$ there is no limit point.

7. Invariant sets. Definitions.

The well known definitions given in the theory of dynamical systems can be applied in a strong and a weak form to the generalized control systems, according to whether it is required that all or only some of the trajectories starting at a set, have a certain behavior.

In all the following definitions, a certain generalized control system is supposed to be given by its attainability functions, $F(x, t_o, t)$ and $G(x, t, t_o)$.

Definition 7.1: The set $A \subset X$ is called strongly invariant, if for all $t_1 \geq t_o$, $F(A, t_o, t_1) \subset A$ and $G(A, t_1, t_o) \subset A$.

Definition 7.2: The set $A \subset X$ is called positively strongly invariant, if for all $t_1 \geq t_o$, $F(A, t_o, t_1) \subset A$.

Definition 7.3: The set $A \subset X$ is called negatively strongly invariant, if for all $t_1 \geq t_o$, $G(A, t_1, t_o) \subset A$.

Remark: It is easily seen that these definitions could be given in the following equivalent way: for every $x \in A$, $F(x, t_o, t_1) \subset A$ and so on.

Definition 7.1 is also equivalent to:

$$F(A, t_o, t_1) = G(A, t_1, t_o) = A.$$

Indeed, $A \subset G(F(A, t_o, t_1), t_1, t_o) \subset G(A, t_1, t_o) \subset A$ gives $A = G(A, t_1, t_o)$ and similarly for $F(A, t_o, t_1)$.

Definition 7.4: The set $A \subset X$ is called weakly invariant, if for all $t_1 \geq t_o$ and all $x \in A$, $F(x, t_o, t_1) \cap A \neq \emptyset$ and $G(x, t_1, t_o) \cap A \neq \emptyset$.

Definition 7.5: The set $A \subset X$ is called positively weakly invariant, if for all $t_1 \geq t_o$ and all $x \in A$, $F(x, t_o, t_1) \cap A \neq \emptyset$.

Definition 7.6: The set $A \subset X$ is called negatively weakly invariant, if for all $t_1 \geq t_o$ and all $x \in A$, $G(x, t_1, t_o) \cap A \neq \emptyset$.

Theorem 7.1. (Barbashin): A necessary and sufficient condition for a closed set A to be positively weakly invariant, is that for any $x_o \in A$ and any t_o, there exists a trajectory $\varphi_o(t)$ defined for $t \in [t_o, \infty)$, starting at $\varphi_o(t_o) = x_o$ and totally contained in A.

The sufficiency is obvious; to prove the necessity, suppose $t_o = 0$. Assuming A is positively weakly invariant, a trajectory through x_o will be constructed, which is contained in A for all $t \geq t_o = 0$.

In the interval $[0, 1]$ consider the following sequence of trajectories.

As $F(x_o, 0, 1) \cap A \neq \emptyset$, there exists a point $x_{11} \in F(x_o, 0, 1) \cap A$ and a trajectory $\varphi_1(t)$ such that $\varphi_1(0) = x_o$, $\varphi_1(1) = x_{11}$.

Similarly it is possible to determine $x_{21} \in F(x_o, 0, \frac{1}{2}) \cap A$ and $x_{22} \in F(x_{21}, \frac{1}{2}, 1) \cap A$, and a trajectory $\varphi_2(t)$ such that $\varphi_2(0) = x_o$, $\varphi_2(\frac{1}{2}) = x_{21}$, $\varphi_2(1) = x_{22}$.

In the same way we determine $\varphi_{n+1}(t)$ such that $\varphi_{n+1}(0) = 0$ and for $t = \frac{p}{2^n}$, $(p = 1,2, \ldots, 2^n)$, $\varphi_{n+1}(t)$ belongs to A.

According to Theorem 6.2 some subsequence of the φ_i converge and define $\varphi_o(t)$ in the interval $[0, 1]$. By construction, $\varphi_o(t)$ belongs to the closed set A for all values of t which are binary fractions, and therefore for all $t \in [0, 1]$.

The same procedure can be used to define $\varphi_o(t)$ in the interval $[1, 2]$, and so on, on the whole real halfline $[0, \infty)$. This proves the theorem.

Theorem 7.2: A necessary and sufficient condition for a compact set A to be negatively weakly invariant is that for any $x_o \in A$ and any t_o, there exists a trajectory $\varphi_o(t)$ defined for $t \in (-\infty, t_1]$, ending at $\varphi_o(t_o) = x_o$ and totally contained in A.

The proof is the same as in the previous theorem, with the exception of the construction of the trajectory $\varphi_o(t)$ in each finite interval, for example $[-1, 0]$; this is due to the fact that the theorem 6.3 cannot be applied directly to prove the convergence of a subsequence of the $\varphi_n(t)$, because $G(x_o, t_o, t)$ is not necessarily compact. This difficulty can be overcome by taking into account that all the $\varphi_n(t)$, $(t \in [-1, 0])$, belong to the compact set $F(A \cap G(x_o, 0, -1), -1, [-1, 0])$. This insures the convergence to some limit trajectory $\varphi_o(t)$.

8. Generalized Liapunov's functions.

Definition 8.1: Given a scalar function $v(x, t)$ defined in an open region $G \subset X \times R$, and a closed set $A \subset X \times R$, the function $v(x, t)$ will be called "positive with respect to the set A" (written: "positive (A)") if:

i) $G \supset A$

ii) $v(x, t)$ is lower semicontinuous, i.e. for any sequences $x_i \to x_o$, $t_i \to t_o$ $(i = 1,2,3, \ldots)$, $\underline{\lim}\, v(x_i, t_i) \geq v(x_o, t_o)$.

iii) $v(x, t) \leq 0$ for $x \in A$.

iv) $v(x, t) > 0$ for $x \notin A$.

Definition 8.2: The scalar function $v(x, t)$ defined in an open region $G \subset X \times R$, is called positive definite with respect to the closed set $A \subset X \times R$ (written "positive definite (A)"), if

i) $G \supset A$

ii) $v(x, t)$ is positive (A);

iii) there exist two continuous and strictly increasing functions $v_1(r)$ and $v_2(r)$ of the real variable $r \geq 0$ such that $v_1(0) = v_2(0) = 0$ and for $x \notin A$, $v_1(r) \leq v(x, t) \leq v_2(r)$ where $r = \rho((x, t), A)$, ρ being the distance in $X \times R$.

<u>Definition 8.4</u>: The scalar function $v(x, t)$ defined in an open region $G \subset X \times R$, is called positive definite with respect to the closed set $A \subset X$, if $v(x, t)$ is positive definite with respect to the set $A \times R$.

<u>Definition 8.5</u>: The upper and lower right total derivatives of the function $v(x, t)$ with respect to the generalized control system characterized by the attainability function $F(x, t, \tau)$ are defined by:

$$\bar{\dot{v}}^+(x, t) = \lim_{\delta \to 0^+} \text{l.u.b.}\left\{\frac{v(F(x,t,\tau),\tau) - v(x,t)}{\tau - t};\quad t < \tau < t + \delta\right\},$$

$$\underline{\dot{v}}^+(x, t) = \lim_{\delta \to 0^+} \text{g.l.b.}\left\{\frac{v(F(x,t,\tau),\tau) - v(x,t)}{\tau - t};\quad t < \tau < t + \delta\right\}.$$

<u>Theorem 8.1</u>: If the set $A \subset X$ is positively weakly invariant, so is its closure $\bar{A}$.

Proof: Assume $x_o \in \bar{A}$, $x_i \in A$, $(i = 1,2,3, \ldots)$ and $x_i \to x_o$. Then, for any t_o, there exist positive half trajectories $\varphi_i(t)$,

$(t \geq t_o)$, such that $\varphi_1(t_o) = x_1$ and $\varphi_1(t) \in A$. By Theorem 6.4 there is a limit trajectory $\varphi_o(t)$; therefore $\varphi_o(t_o) = x_o$, $\varphi_o(t) \in \bar{A}$ and by Theorem 7.1 $\bar{A}$ is positively weakly invariant.

<u>Theorem 8.2</u>: If $A \subset X$ is closed, $v(x, t)$ is positive (A) and for all x belonging to the boundary of A and all t, $\bar{v}^+(x, t) < 0$, then A is strongly positively invariant.

Proof: Assuming the contrary, there are some $y_o \in A$, $t_o < \tau$ and a trajectory $\varphi(t)$ such that $\varphi(t_o) = y_o$, $\varphi(\tau) \notin A$. Define $\tau_o = \text{l.u.b.}\{\tau;\ \varphi([t_o, \tau]) \subset A\}$. Then $x_o = \varphi(\tau_o) \in A$ because A is closed, but there is a sequence $x_n = \varphi(\tau_n) \notin A$ with $\tau_n \to \tau_o^+$ $(n = 1,2,3, \ldots)$.

Therefore x_o belongs to the boundary of A, $v(x_o, t) \leq 0$, but $v(x_n, t') > 0$ for any t, t' and

$$\overline{\lim} \frac{v(x_n, t') - v(x_o, t)}{t' - t} \geq 0, \quad \text{for any} \quad t' \to t$$

which contradicts $\bar{v}^+(x_o, t) < 0$.

<u>Theorem 8.3</u>: If $A \subset X$ is closed and positively strongly invariant, and if $v(x, t)$ is positive (A), then $\bar{v}^+(x, t) \leq 0$ holds for any (x, t) such that $v(x, t) = 0$.

Proof: If $v(x, t) = 0$, then $x \in A$ and $F(x, t, \tau) \in A$, $(\tau > t)$, so that

$$\text{l.u.b.}\ \frac{v(F(x, t, \tau), \tau) + v(x, t)}{\tau - t} \leq 0.$$

The proof follows then from the definition of $\bar{\dot{v}}^+(x, t)$.

Theorem 8.4: If $A \subset X$ is closed, $v(x, t)$ is positive (A) and for all x belonging to the boundary of A and all t, $\underline{\dot{v}}^+(x, t) < 0$, then A is weakly positively invariant.

Proof: Assume the contrary. Then there are $y \in A$, $t_o \leq \tau_1$ such that $F(y, t_o, \tau_1) \cap A = \emptyset$. As both sets are closed, there is some neighborhood of $F(y, t_o, \tau_1)$ which does not meet A. By continuity of F, there is an interval such that $F(y, t_o, \tau) \cap A = \emptyset$ for $\tau_o' < \tau \leq \tau_1$. Suppose τ_o is the minimum of the values τ_o' satisfying this condition (this value τ_o exists and $\tau_o \geq t_o$). Then $F(y, t_o, \tau_o) \cap A \neq \emptyset$. Let $x \in F(y, t_o, \tau_o) \cap A$. Then $x \in \partial A$ and $v(x, \tau_o) \leq 0$. But for $\tau_o < \tau \leq \tau_1$, $F(x, \tau_o, \tau) \cap A = \emptyset$ and $v(x', t) > 0$ for any t and $x' \in F(x, \tau_o, \tau)$. Therefore

$$\frac{v(F(x, \tau_o, \tau), \tau) - v(x, \tau_o)}{\tau - \tau_o} > 0$$

for $\tau_o < \tau \leq \tau_1$, which contradicts the hypothesis $\underline{\dot{v}}^+(x, \tau_o) < 0$.

Theorem 8.5: If $A \subset X$ is closed and positively weakly invariant, if $v(x, t)$ is positive (A), then $\underline{\dot{v}}^+(x, t) \leq 0$ holds for any (x, t) such that $v(x, t) = 0$.

Proof: If $v(x, t) = 0$, then $x \in A$ and for any $\tau \geq t$, $F(x, t, \tau) \cap A \neq \emptyset$. Therefore

$$\text{g.l.b.}\left(\frac{v(F(x, t, \tau), \tau) - v(x, t)}{\tau - t}\right) \leq 0$$

and $\underline{\dot{v}}^+(x, t) < 0$ follows.

__Remark__: Theorems 8.3 and 8.5 are really existence theorems for Liapunov functions, because functions $v(x, t)$ which are positive (A) always exist, for example $v(x) = \rho(x, A)$.

9. __Stability__.

Many different kinds of stability are well known from the theory of dynamical systems. In this paper only the most important types will be considered, rather as an example than as a full development of the theory. The definitions are given according to Yoshizawa [12]; they refer to a certain generalized control system characterized by the function $F(x, t, \tau)$.

__Definition 9.1__: The set $A \subset X$ is said to be __uniformly strongly stable__, if for any $\epsilon > 0$ there is a $\delta > 0$ such that $F(S_\delta(A), t, \tau) \subset S_\epsilon(A)$ for all $t \leq \tau$.

__Remark__: If A is uniformly strongly stable, then $\bar{A}$ is strongly positively invariant and uniformly strongly stable.

Indeed, if $x_o \in \bar{A}$, $\rho^*(F(x_o, t, \tau), \bar{A}) = 0$ and therefore $F(x_o, t, \tau) \subset \bar{A}$.

<u>Definition 9.2</u>: The set $A \subset X$ is said to be <u>uniformly weakly stable</u> if for any $\epsilon > 0$ there is a $\delta > 0$ such that if $\rho(x_o, A) < \delta$, $t_o \in R$, there exists a trajectory $\varphi(t)$ through (x_o, t_o), (i.e. $\varphi_o(t_o) = x_o$) satisfying $\rho(\varphi_o(t), A) < \epsilon$ for all $t \geq t_o$.

<u>Remark</u>: If A is uniformly weakly stable, then $\bar{A}$ is positively weakly invariant.

<u>Definition 9.3</u>: The set $A \subset X$ is said to be <u>uniformly strongly quasi-asymptotically stable</u>, if for some fixed $\delta > 0$, and for any $\epsilon > 0$, there is a $T(\epsilon) > 0$ such that for all $t \geq t_o + T(\epsilon)$, $F(S_\delta(A), t_o, t) \subset S_\epsilon(A)$.

<u>Definition 9.4</u>: The set $A \subset X$ is said to be <u>uniformly weakly quasi-asymptotically stable</u>, if for some fixed $\delta > 0$, given any pair $(x_o, t_o) \in S_\delta(A) \times R$ there is a trajectory $\varphi(t)$, with $\varphi(t_o) = x_o$, such that for any $\epsilon > 0$ there exists a $T(\epsilon) > 0$ with the property that $\rho(\varphi(t), A) < \epsilon$ for all $t \geq t_o + T(\epsilon)$, $T(\epsilon)$ being independent from x_o, t_o.

<u>Definition 9.5</u>: The set $A \subset X$ is said to be <u>uniformly strongly asymptotically stable</u> if it is uniformly strongly stable and uniformly strongly quasi-asymptotically stable.

<u>Definition 9.6</u>: The set $A \subset X$ is said to be <u>uniformly weakly asymptotically stable</u>, if it is uniformly weakly stable and uniformly weakly quasi-asymptotically stable.

The following lemma concerning functions of a real variable will be needed.

<u>Lemma 9.1</u>: If the real function of the real variable $u(t)$, defined in the interval $[0, \epsilon]$, is lower semi-continuous, and at every point of that interval the right lower derivative satisfies

$$\underline{\dot{u}}^{+}(t) \leq 0,$$

then $u(T) \leq u(0)$.

Proof: Suppose $u(0) = 0$; it will be proven that for any given $\epsilon > 0$, $u(T) \leq \epsilon$.

Indeed, there is some value t_1 such that $0 < t_1 < T$ and

$$\frac{u(t_1)}{t_1} < \frac{\epsilon}{T}.$$

This follows from the fact that $\underline{\lim}_{t \to 0^+} \frac{u(t)}{t} \leq 0$. There is also a value t_2 such that $t_1 < t_2 < T$ and

$$\frac{u(t_2) - u(t_1)}{t_2 - t_1} < \frac{\epsilon}{T}.$$

In this way a sequence $t_1, t_2, t_3, \ldots$ can be obtained, such that

$$u(t_n) = u(t_n) - u(t_{n-1}) + u(t_{n-1}) - u(t_{n-2}) + \ldots$$
$$\ldots + u(t_2) - u(t_1) + u(t_1) - u(0) <$$
$$< \frac{\epsilon}{T}(t_n - t_{n-1} + t_{n-1} - t_{n-2} + \ldots + t_1 - 0) = \frac{\epsilon}{T} t_n.$$

If by this procedure the value $t = T$ can be indefinitely approached, i.e. $t_n \to T$, then by the lower semicontinuity

$$u(T) \le \underline{\lim}\, u(t_n) \le \frac{\epsilon}{T} \lim t_n = \epsilon.$$

In case T cannot be approached indefinitely, there is a g.l.b. of those values of t which cannot be passed by any such sequence t_n. Call τ this g.l.b. Then there is a sequence $t_n \to \tau$ and

$$u(\tau) \le \frac{\epsilon}{T}\tau.$$

But $\underline{\dot u}^+(\tau) \le 0$ and therefore some τ' exists, such that $\tau < \tau' < T$ and $\dfrac{u(\tau') - u(\tau)}{\tau' - \tau} < \dfrac{\epsilon}{T}$, so that the sequence can be extended farther than τ, against our assumption. Therefore it can be extended approaching T indefinitely.

<u>Corollary</u>: If $u(t)$ is lower semi-continuous in $[0, T]$, and $\overline{\dot u}^+(t) \le 0$, then $u(T) \le u(0)$.

<u>Lemma 9.2</u>: If $v(x)$ is a real, lower semi-continuous function defined on the space X, and if with respect to a generalized control system, $\overline{\dot v}^+(x) \le 0$, then the set $A(\lambda) = \{x;\ v(x) \le \lambda\}$, supposedly non-empty, is a closed positively strongly invariant set.

Proof: $A(\lambda)$ is closed because $v(x)$ is lower semi-continuous. Assume $x \in A(\lambda)$, but for some $t_o < t_1$, $y \in F(x, t_o, t_1)$ does not belong to $A(\lambda)$. Then there exists a trajectory $\varphi(t)$ from $x = \varphi(t_o)$ to $y = \varphi(t_1)$. Along $\varphi(t)$

$$\frac{\overline{d}^{+}}{dt} v(\varphi(t)) = \overline{\lim_{\delta \to 0^+}} \frac{v(\varphi(t+\delta)) - v(\varphi(t))}{\delta} \le \bar{\dot{v}}^{+}(x) \le 0.$$

Besides, $v(\varphi(t))$ is lower semi-continuous, so by the corollary of Lemma 9.1, $v(y) \le v(x) \le \lambda$ contradicting our assumption. This proves that for any $x \in A(\lambda)$ and $t_o \le t$, $F(x, t_o, t) \subset A(\lambda)$.

The same proof applies to the following.

__Lemma 9.3__: If $v(x, t)$ is a real, lower semi-continuous function defined on the space $X \times R$ and if $\bar{\dot{v}}(x, t)^{+} \le 0$, then the set $A(\lambda) = \{(x, t); v(x, t) \le \lambda\}$, supposed non-empty, is a closed positively strongly invariant set.

__Lemma 9.4__: If $v(x)$ is a real lower semi-continuous function defined on the space X, and if $\underline{\dot{v}}^{+}(x) = 0$, then the set $A(\lambda) = \{x; v(x) \le \lambda\}$, supposed non-empty, is a closed positively weakly invariant set.

Proof: Suppose $\lambda = 0$, $A(0) = A = \{x; v(x) \le 0\}$. Taking any fixed x_o, t_o, the function

$$u(t) = \text{g.l.b.}\{v(x); x \in F(x_o, t_o, t)\}$$

is lower semi-continuous in t. Indeed, as $F(x_o, t_o, t)$ is compact, the g.l.b. is really the minimum of $v(x)$. As $v(x)$ is lower semi-continuous in x, given $\epsilon > 0$, for each x there is a $\delta(\epsilon, x) > 0$

such that for $\rho(x, x') < \delta$, $v(x') \geq v(x) - \epsilon$. $F(x_o, t_o, t)$ can be covered by a finite number of such neighborhoods, therefore there is an $\eta(\epsilon)$ such that for $\rho(x', F(x_o, t_o, t)) < \eta$, $v(x') \geq \min v(x) - \epsilon = u(t) - \epsilon$. As $F(x_o, t_o, t)$ is continuous in t, the lower semi-continuity of $u(t)$ follows.

In order to evaluate the right lower derivative of $u(t)$ at the point t_1, suppose that $v(x)$ attains at the point x_1 its minimum in $F(x_o, t_o, t_1)$:

$$u(t_1) = v(x_1) = \min\{v(x);\ x \in F(x_o, t_o, t_1)\}.$$

As $\underline{\dot{v}}^+(x_1) \leq 0$, there are sequences τ_i and y_i, $(i = 1,2,3, \ldots)$ such that $\tau_i \to t_1^+$, $y_i \in F(x_1, t_1, \tau_i)$ and

$$\lim_{i \to \infty} \frac{v(y_i) - v(x_1)}{} = \alpha = \underline{\dot{v}}^+(x_1) \leq 0.$$

Therefore,

$$\underline{\dot{u}}^+(t_1) = \lim_{\theta \to 0^+} \text{g.l.b.}\{\frac{u(\tau) - u(t_1)}{\tau - t_1};\ 0 < \tau < \theta\} \leq$$

$$\leq \lim_{i \to \infty} \frac{u(\tau_i) - u(t_1)}{\tau - t_1} \leq$$

$$\leq \lim_{i \to \infty} \frac{v(y_i) - v(x_1)}{\tau - t_1} = \alpha \leq 0,$$

proving that $\underline{\dot{u}}^+(t) \leq 0$. Hence, Lemma 9.1 can be applied and for all $t > t_o$, and

$$u(t) = \min\{v(x);\ x \in F(x_o, t_o, t)\} \leq u(t_o).$$

Therefore $F(x_o, t_o, t) \cap A \neq \emptyset$, proving the lemma.

Taking the space $X \times R$ as phase-space, the following lemma is obtained.

Lemma 9.5: If $v(x, t)$ is a real, lower semi-continuous function defined on $X \times R$, and if $\underline{\dot{v}}^+(x, t) \leq 0$, then the set $A(\lambda) = \{(x, t);\ v(x, t) \leq \lambda\}$, supposed non-empty, is a closed positively weakly invariant set.

Remark: In the preceding lemmas, the function $v(x)$ (respect. $v(x, t)$) does not need to be defined on the whole space X (respect. $X \times R$) but on a domain G such that $A(\lambda)$ belongs to the interior of G.

Lemma 9.6: If $v(x, t)$ is a real, lower semi-continuous function defined on a closed set $D \subset X \times R$, and if $\underline{\dot{v}}^+(x, t) \leq 0$, if $A = \{(x, t);\ v(x, t) \leq \lambda\}$ is a non-empty subset of D, then for each $(x_o, t_o) \in A$ there is a trajectory $\varphi(t)$ such that $\varphi(t_o) = x_o$ and one of the following two possibilities holds:

i) $\varphi(t) \in A$ for any $t > t_o$; or

ii) $\varphi(t)$ leaves A at a point belonging to the boundary of D.

Indeed if for each t in the interval $[t_o, t_1]$, the set $\{F(x_o, t_o, t), t\} \cap A$ is non-empty, then the construction used in Theorem 7.1 proves the existence of a trajectory $\varphi(t)$ starting at (x_o, t_o) and contained in A for $t_o \leq t \leq t_1$. If this trajectory cannot be extended in A, there are values $\tau_1 \to t_1^+$ such that

$$\{F(x_o, t_o, \tau_1), \tau_1\} \cap A = \emptyset.$$

Hence, $\varphi(t_1)$ belongs to the boundary of A. Assuming that it does not belong to the boundary of D, then for all τ_1 sufficiently near t_1,

$$\{F(\varphi(t_1), t_1, \tau_1), \tau_1\} \subset D.$$

In this case the procedure used in Lemma 9.4 can be applied and

$$u(\tau_1) = \min\{v(x, \tau_1); \ (x, \tau_1) \in (F(\varphi(t_1), t_1, \tau_1), \tau_1)\} \leq u(t_1).$$

This is a contradiction, because then

$$\{F(\varphi(t_1), t_1, \tau_1), \tau_1\} \cap A \neq \emptyset.$$

Therefore $(\varphi(t_1), t_1)$ must belong to the boundary of D which proves the lemma.

<u>Theorem 9.1:</u> If $A \subset X$ is closed and $v(x, t)$ is a real function defined in some neighborhood of A, if $v(x, t)$ is positive definite (A) and $\dot{\bar{v}}^+(x, t) \leq 0$, then A is uniformly strongly stable.

Proof: By definition of positive definite function, there exist two strictly increasing continuous functions $v_1(r)$, $v_2(r)$ such that $v_1(0) = v_2(0) = 0$ and for $x \notin A$,

$$v_1(\rho(x, A)) \leq v(x, t) \leq v_2(\rho(x, A)).$$

Given a sufficiently small $\epsilon > 0$. there is $\delta(\epsilon)$ such that $v_2(\delta) = v_1(\epsilon)$; indeed, $v_2(0) = 0$, $v_2(\epsilon) \geq v_1(\epsilon)$ and v_2 is continuous, so that $0 < \delta(\tau) \leq \epsilon$ (fig. 8).

For $\rho(x_o, A) < \delta$ and any t_o, $v(x_o, t_o) \leq v_2(\delta) = v_1(\epsilon)$. By Lemma 9. , the set $\{(x, t); v(x, t) \leq v_1(\epsilon)\}$ is positively strongly invariant, so that for any $t \geq t_o$ and $x \in F(x_o, t_o, t)$, $v(x, t) \leq v_1(\epsilon)$. Therefore

$$v_1(\rho(x, A)) \leq v(x, t) \leq v_1(\epsilon)$$

implies $\rho(x, A) \leq \epsilon$, which proves that

$$F(x_o, t_o, t) \in S_\epsilon(A).$$

<u>Theorem 9.2</u>: If $A \subset X$ is closed and $v(x, t)$ is a real function defined in some neighborhood of A, if $v(x, t)$ is positive definite (A) and $\underline{\dot{v}}^+(x, t) \leq 0$, then A is uniformly weakly stable.

Proof: As in the preceding theorem, there are $v_1(\rho(x, A)) \leq v(x, t) \leq v_2(\rho(x, a))$, and for sufficiently small $\epsilon > 0$, there is $\delta(\epsilon)$ such that $v_2(\delta) = v_1(\epsilon)$.

By Lemma 9.5, the set $B = \{(x, t); v(x, t) \leq v_1(\epsilon)\}$ is closed and positively weakly invariant. By Theorem 7.1, for any x_o, t_o such that $v(x_o, t_o) \in B$, there exists a trajectory $\varphi(t)$ such that $\varphi(t_o) = x_o$ and for any $t \in [t_o, \infty)$, $\varphi(t) \in B$. Therefore, writing $x = \varphi(t)$,

$$v_1(\rho(x, A)) \leq v(x, t) \leq v_1(\epsilon),$$

or

$$\rho(x, A) \leq \epsilon$$

which proves the theorem.

Lemma 9.7: If $u(t)$, $\varphi(t)$ are real functions of a real variable defined in the interval $[0, T]$, if $u(t)$ is lower semi-continuous and $\varphi(t)$ is differentiable, and if $\underline{\dot{u}}^+(t) \leq \frac{d\varphi(t)}{dt}$, then

$$u(T) - \varphi(T) \leq u(0) - \varphi(0).$$

This is an application of Lemma 9.1 to the function $u(t) - \varphi(t)$. Similarly:

<u>Lemma 9.8</u>: If, under the same conditions, $\bar{\dot{u}}^+(t) \leq \frac{d\varphi(t)}{dt}$ then $u(T) - \varphi(T) \leq u(0) - \varphi(0)$.

<u>Theorem 9.3</u>: If $A \subset X$ is closed and $v(x, t)$ is a real function defined in some closed neighborhood $\overline{S_\delta(A)}$, if $v(x, t)$ and $-\bar{\dot{v}}^+(x, t)$

are positive definite (A), then A is uniformly strongly asymptotically stable.

Proof: By Theorem 9.1, A is uniformly strongly stable, so that only the quasi-asymptotic stability has to be proved.

By definition of a positive definite function, there are increasing continuous functions v_1, v_2, w_1, w_2 such that $v_1(0) = v_2(0) = w_1(0) = w_2(0) = 0$ and for $x \notin A$,

$$v_1(\rho(x, a)) \leq v(x, t) \leq v_2(\rho(x, A)),$$

$$w_1(\rho(x, a)) \leq - \bar{\dot{v}}^{+} (x, t) \leq w_2(\rho(x, a)).$$

Besides, it may be assumed that

$$v_1(\delta) = v_2(\delta') = a$$

is finite.

Given any sufficiently small $\epsilon > 0$, the numbers $\delta_1 < 0$, $b > 0$ and $T(\epsilon) > 0$ can be found such that

$$v_2(\delta_1) = v_1(\epsilon),$$

$$w_1(\delta_1) = b$$

and

$$T = T(\epsilon) = \frac{a}{b}.$$

The proof of this theorem will consist in showing that for any $x_o \in S_{\delta'}(A)$ and any t_o, t_1 such that $t_1 > t_o + T(\epsilon)$, every trajectory $\varphi(t)$ starting at $\varphi(t_o) = x_o$, satisfies $\rho(\varphi(t_1), A) < \epsilon$. For every trajectory $x = \varphi(t)$, the inequality $v(x, t) \leq v_2(\delta') = v_1(\delta)$ insures that $\varphi(t)$ remains in $S_\delta(A)$ for $t \geq t_o$; besides

$$\left(\overline{\frac{d}{dt}}\, v(\varphi(t), t)\right)^+ \leq \bar{\dot{v}}^+(x, t) \leq - w_1(\rho(x, A)).$$

If in the interval $[t_o, t_o + T]$, the trajectory $\varphi(t)$ would remain outside $S_{\delta_1}(A)$, so that $\rho(\varphi(t), A) > \delta_1$,

$$\left(\overline{\frac{d}{dt}}\, v(\varphi(t), t)\right)^+ \leq - w_1(\rho(\varphi(t), A)) \leq - w_1(\delta_1) = - b.$$

By Lemma 9.8

$$v(\varphi(t_o + T), t_o + T) \leq a - bT,$$

a being an upper bound of the value $v(\varphi(t_o), t_o)$. Substituting the value of T, the contradiction

$$v(\varphi(t_o + T), t_o + T) \leq a - b \cdot \frac{a}{b} = 0$$

is obtained. Therefore every trajectory starting at x_o, $t_o \in S_{\delta'}(A)$ comes into $S_{\delta_1}(A)$ at some t' in the interval $[t_o, t_o + T]$. But then, for any $t_1 > t'$,

$$v_1(\rho(\varphi(t_1), A)) \leq v(\varphi(t_1), t_1) \leq v(\varphi(t'), t') \leq v_2(\delta_1) = v_1(\epsilon)$$

so that $\rho(\varphi(t_1), A) \leq \epsilon$.

Lemma 9.9: If $v(x, t)$ is a real function defined in $X \times R$, if $f(t)$ is continuously differentiable, and $u(x, t) = v(x, t) \cdot f(t)$, then

$$\underline{\dot{u}}^+(x, t) = \underline{\dot{v}}^+(x, t) \cdot f(t) + v(x, t) \cdot \frac{df(t)}{dt}.$$

Proof:

$$\underline{\dot{u}}^+(x,t) = \lim_{\delta \to 0^+} \text{g.l.b.} \left\{ \frac{v(F(x,t,\tau),\tau) - v(x,t)}{\tau - t};\ t < \tau < t + \delta \right\} =$$

$$= \lim_{\delta \to 0^+} \text{g.l.b.} \left\{ \frac{v(F(x,t,\tau),\tau) \cdot f(\tau) - v(x,t) \cdot f(t)}{t - \tau};\ t < \tau < t + \delta \right\} =$$

$$= \lim_{\delta \to 0^+} \text{g.l.b.} \left\{ \frac{[v(F(x,t,\tau),\tau) - v(x,t)]f(\tau)}{\tau - t} + \frac{v(x,t)(f(\tau) - f(t))}{\tau - t} \right\} =$$

$$= \lim_{\delta \to 0^+} \text{g.l.b.} \left\{ \frac{v(F(x,t,\tau),\tau) - v(x,t)}{\tau - t};\ t < \tau < t + \delta \right\} \cdot f(t) +$$

$$+ v(x,t) \frac{df(t)}{dt} =$$

$$= \underline{\dot{v}}^+(s, t) \cdot f(t) + v(x, t) \cdot \frac{df(t)}{dt}.$$

Theorem 9.4: If $A \subset X$ is closed and $v(x, t)$ is a real function defined and uniformly bounded in some closed neighborhood $\overline{S_\delta(A)}$, if $v(x, t)$ and $-\underline{\dot{v}}^+(x, t)$ are positive definite (A), then A is uniformly weakly asymptotically stable.

Proof: By Theorem 9.2, A is uniformly weakly stable, so that only the quasi-asymptotic stability has to be proved.

By definition of a positive definite function, there are increasing continuous functions v_1, v_2, w_1, w_2 such that $v_1(0) = v_2(0) = w_1(0) = w_2(0) = 0$ and for $x \notin A$,

$$v_1(\rho(x, A)) \leq v(x, t) \leq v_2(\rho(x, a)),$$

$$w_1(\rho(x, A)) \leq - \underline{\dot{v}}^+(x, t) \leq w_2(\rho(x, A)).$$

Besides, it may be assumed that

$$v_1(\delta) = v_2(\delta') = a$$

is finite.

Given $\epsilon > 0$, the existence of a number T will be found such that for any $(x_o, t_o) \in S_{\delta'}(A) \times R$, there is a trajectory $\varphi(t)$ such that

i) $\varphi(t_o) = x_o$,

ii) $\varphi(t_o + t_1) \in S_\eta(A)$ for some $t_1 \leq T$, where

iii) $v_2(\eta) = v_1(\epsilon)$.

The function $\dfrac{w_1(\rho)}{v_2(\rho)}$ is continuous and positive in the interval $[\eta, \delta]$, hence, in that interval it has a lower bound $\alpha = \alpha(\eta) > 0$. Now the function

$$u(x, t) = v(x, t) \cdot e^{\alpha(t - t_o)}$$

is lower semi-continuous and for $\eta \leq \rho(x, A) \leq \delta$,

$$\underline{\dot{u}}^+(x, t) = [\underline{\dot{v}}^+(x, t) + \alpha v(x, t)]e^{\alpha(t - t_o)} \leq$$

$$\leq [-v_1(\rho(x, A)) + \alpha v_2(\rho(x, A))]e^{\alpha(t - t_o)} =$$

$$= [-\frac{v_1(\rho(x, A))}{v_2(\rho(x, A))} + \alpha]v_2(\rho(x, a))e^{\alpha(t - t_o)} \leq 0.$$

Now the Lemma 9.6 can be applied to the function $u(x, t)$, taken on the set $D = \{(x, t); \eta \leq \rho(x, A) \leq \delta\}$, where $\underline{\dot{u}}^+(x, t) \leq 0$.

Suppose (x_o, t_o) fixed, $x_o \in S_{\delta'}(A)$. Hence, $v(x_o, t_o) \leq v_2(\rho(x_o, A)) < v_2(\delta')$. Now, taking $v_2(\rho(x_o, A)) < k < v_2(\delta')$, the set

$$B = \{(x; t); u(x, t) \leq k, t \geq t_o, \rho(x, A) \geq \eta\}$$

satisfies the requirements of Lemma 9.6. Therefore there is a trajectory $\varphi(t)$ starting at $\varphi(t_o) = x_o$ and:

i) remaining in B for all $t > t_o$, or

ii) leaving B at the boundary of D.

But $\varphi(t)$ must leave the set B for some $t_1 \leq t_o + T$, where

$$T > \frac{1}{\alpha} \log \frac{v_2(\delta')}{v_1(\eta)},$$

because for $t > t_o + T$ the section of B with the plane $t = T$ is empty. Indeed

$$u(x,\ t) \le k < v_2(\delta') \quad \text{and}$$

$$u(x,\ t) = v(x,\ t)e^{\alpha T} > (x,\ t)\exp(\alpha \cdot \frac{1}{\alpha}\log \frac{v_2(\delta')}{v_1(\eta)} \ge$$

$$\ge v_1(\rho(x,\ a))\ \frac{v_2(\delta')}{v_1(\eta)} \ge v_2(\delta')$$

are contradicting. Therefore $\varphi(t)$ must leave the set B on the part of the boundary where $\rho(x,\ A) = \eta$ or $\rho(x,\ A) = \delta$. But B does not meet the set $\{x;\ \rho(x,\ A) = \delta\}$, because on this set

$$u(x,\ t) = v(x,\ t)e^{\alpha(t - t_o)} \ge v_1(\delta) = v_2(\delta').$$

Hence, $\varphi(t)$ leaves B through the boundary where $\rho(x,\ A) = \eta$ and $\varphi(t_1) \in S_\eta(A)$ for some $t_1 \le t_o + T$, as it was to be proved. Note that T does not depend on x_o, t_o but only on η defined by $v_2(\eta) = v_1(\epsilon)$.

Therefore the existence of the trajectory $\varphi(t)$, starting at $(x_o,\ to)$ and such that for $t_1 \in [t_o,\ t_o + T]$, $\varphi(t_1) \in S_\eta(A)$, has been proved. In order to continue this trajectory for $t > t_1$, the same procedure can be applied, taking $\frac{\epsilon}{2}$ instead of ϵ. So, $\varphi(t)$ will be extended to some t_2 where $\varphi(t_2) \in S_{\eta_2}(A)$, where $v_2(\eta_2) = v_1(\frac{\epsilon}{2})$. In general, after n steps, $\varphi(t_n) \in S_{\eta_n}(A)$, where $v_2(\eta_n) = v_1(\frac{\epsilon}{n})$. So, $\varphi(t)$ is defined for every $t \ge t_o$.

Besides, for $t \geq t_1$,

$$v_1(\rho(\varphi(t), A)) \leq v(\varphi(t), t) \leq v_2(\eta) = v_1(\epsilon)$$

so that $\rho(\varphi(t), A) \leq \epsilon$. Similarly, for $t \geq t_n$, $\rho(\varphi(t), A) \leq \frac{\epsilon}{n}$. This proves that $\rho(\varphi(t), A) \to 0$ for $t \to \infty$. The fact that the $t_n - t_o$ are bounded in terms of the $T(\eta)$, gives the uniformity needed for the definition of uniform quasi-asymptotic stability. Therefore the theorem is proved.

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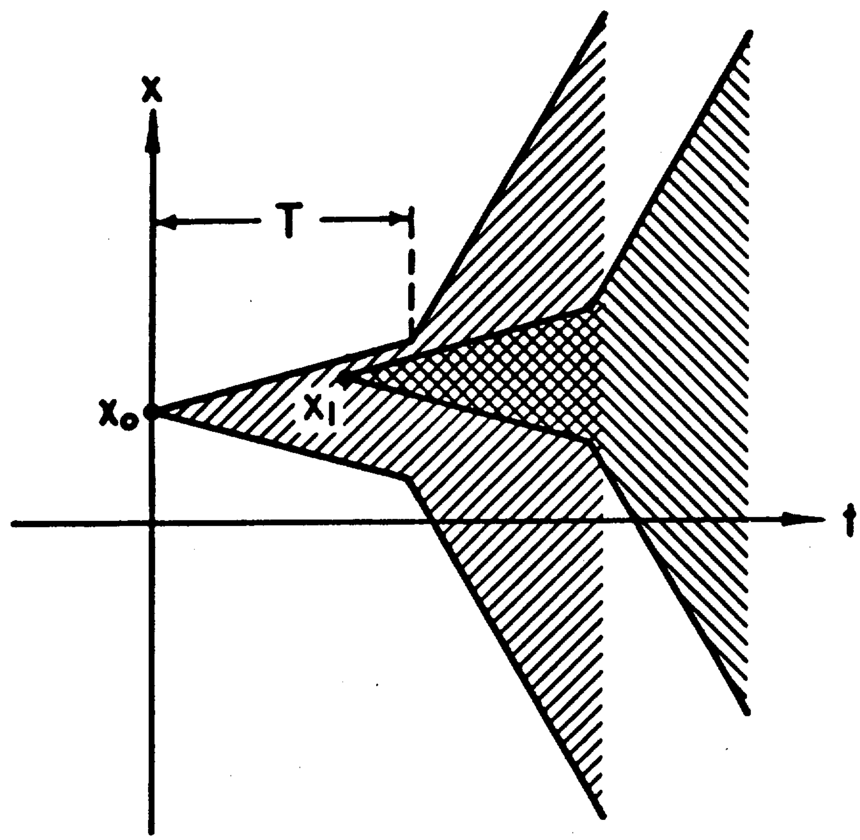

Fig. 1

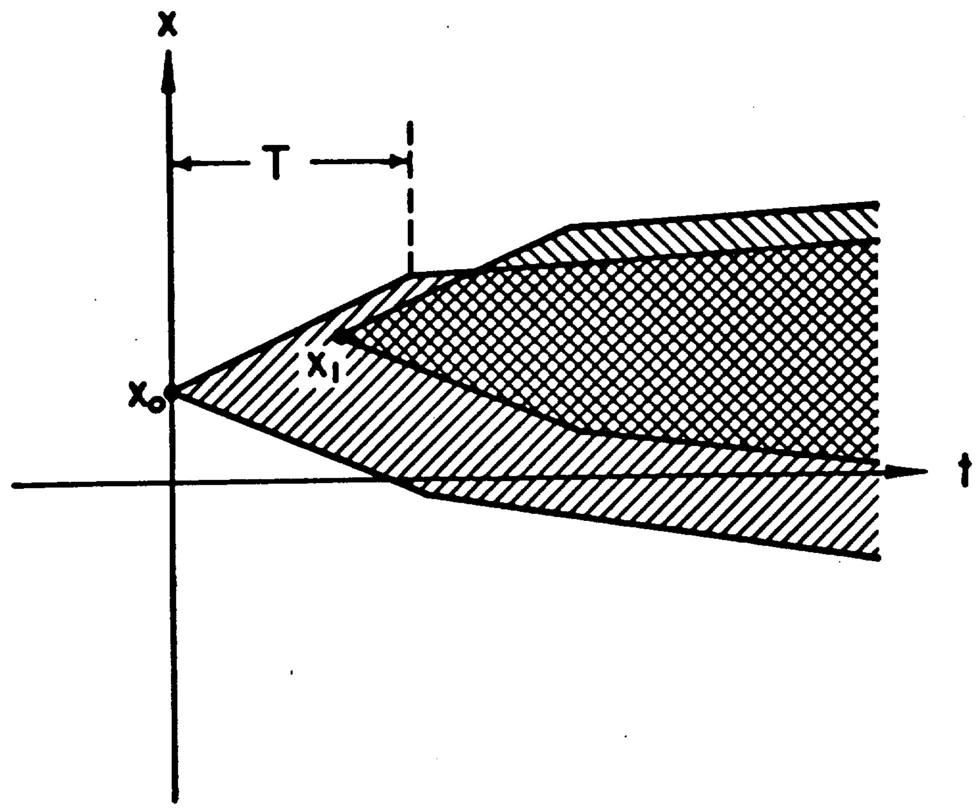

Fig. 2

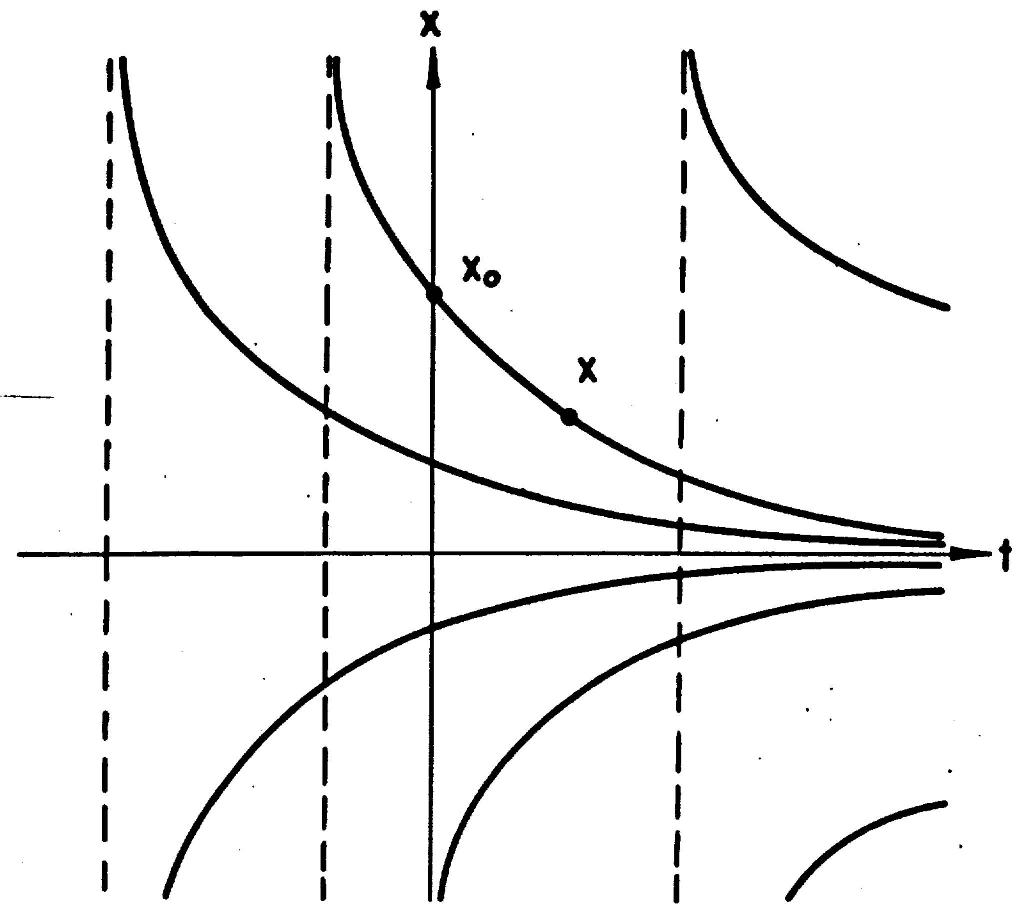

Fig. 3

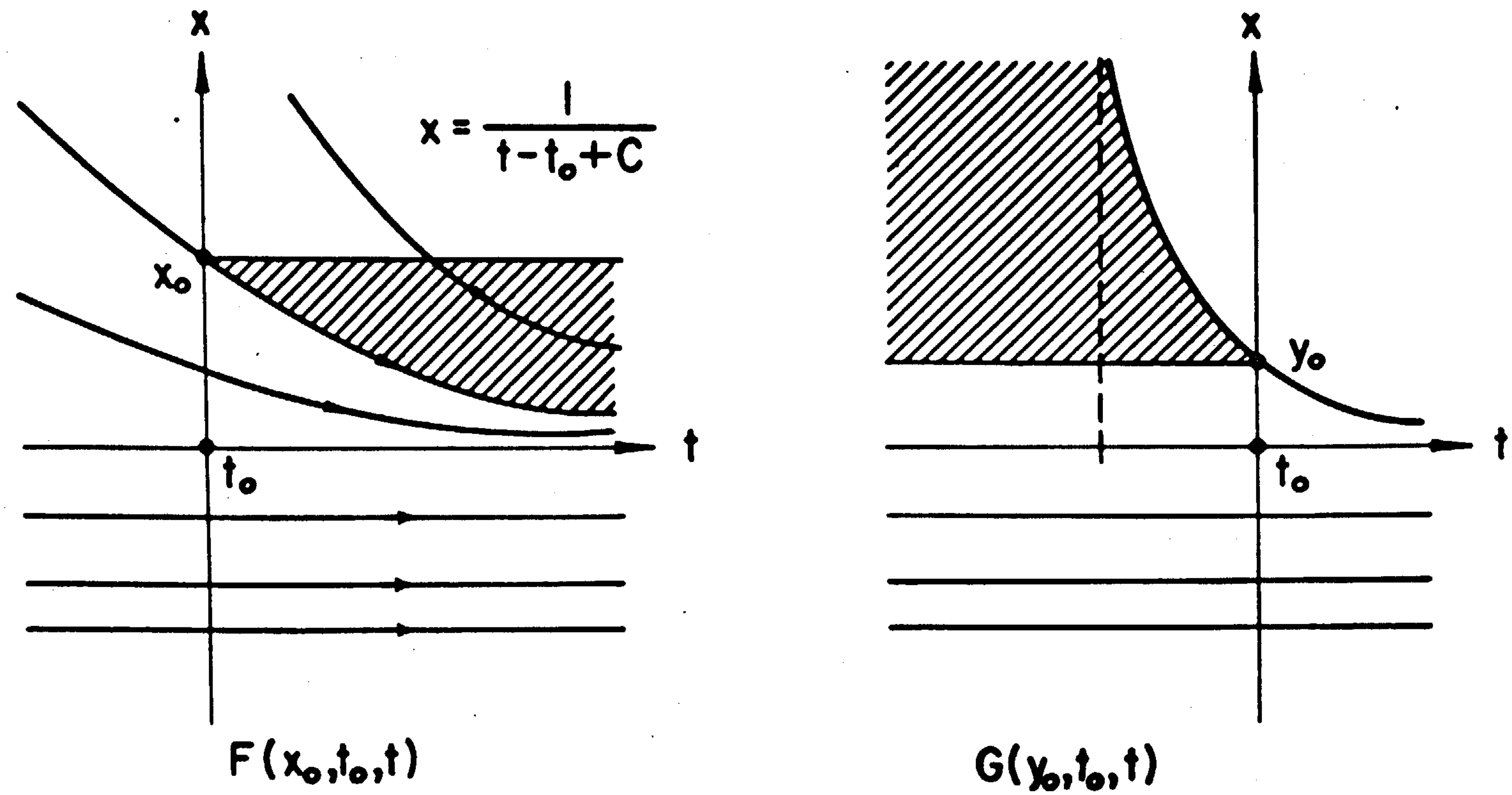

Fig. 4

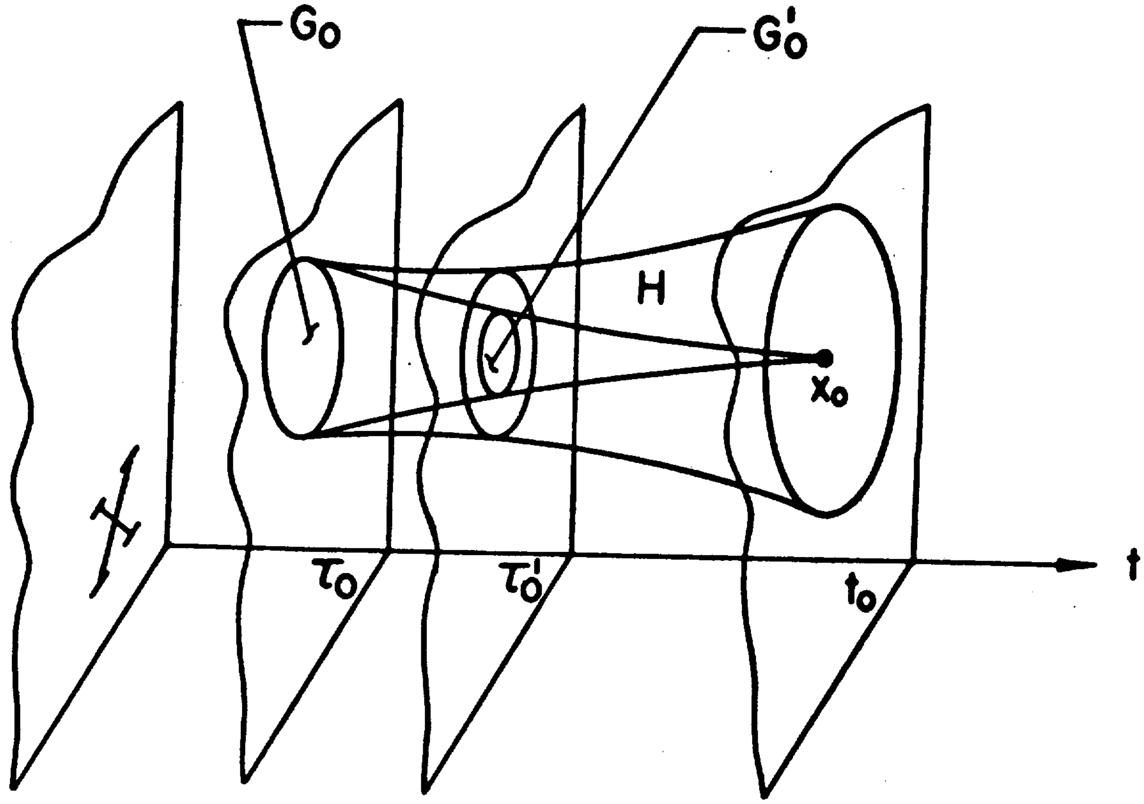

Fig. 5

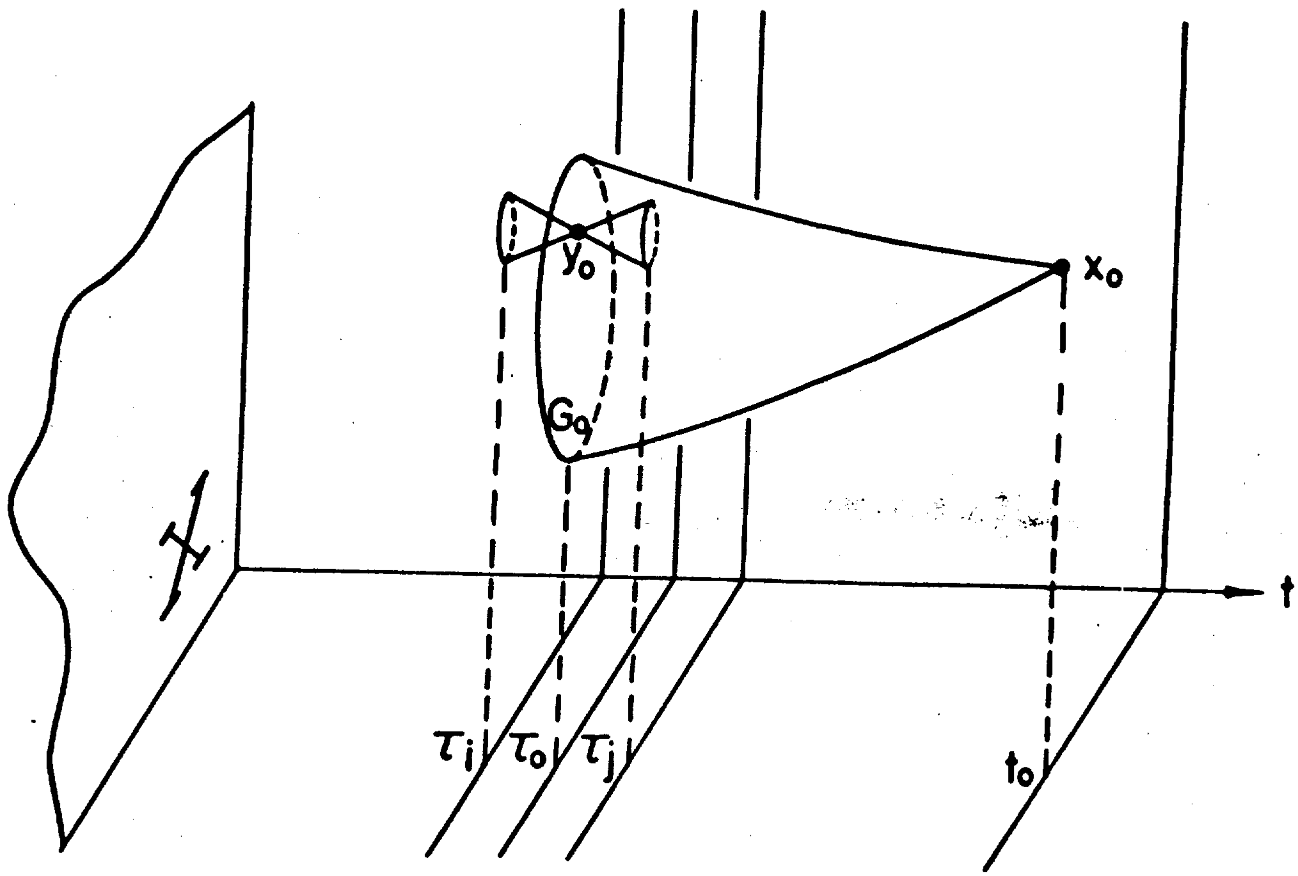

Fig. 6

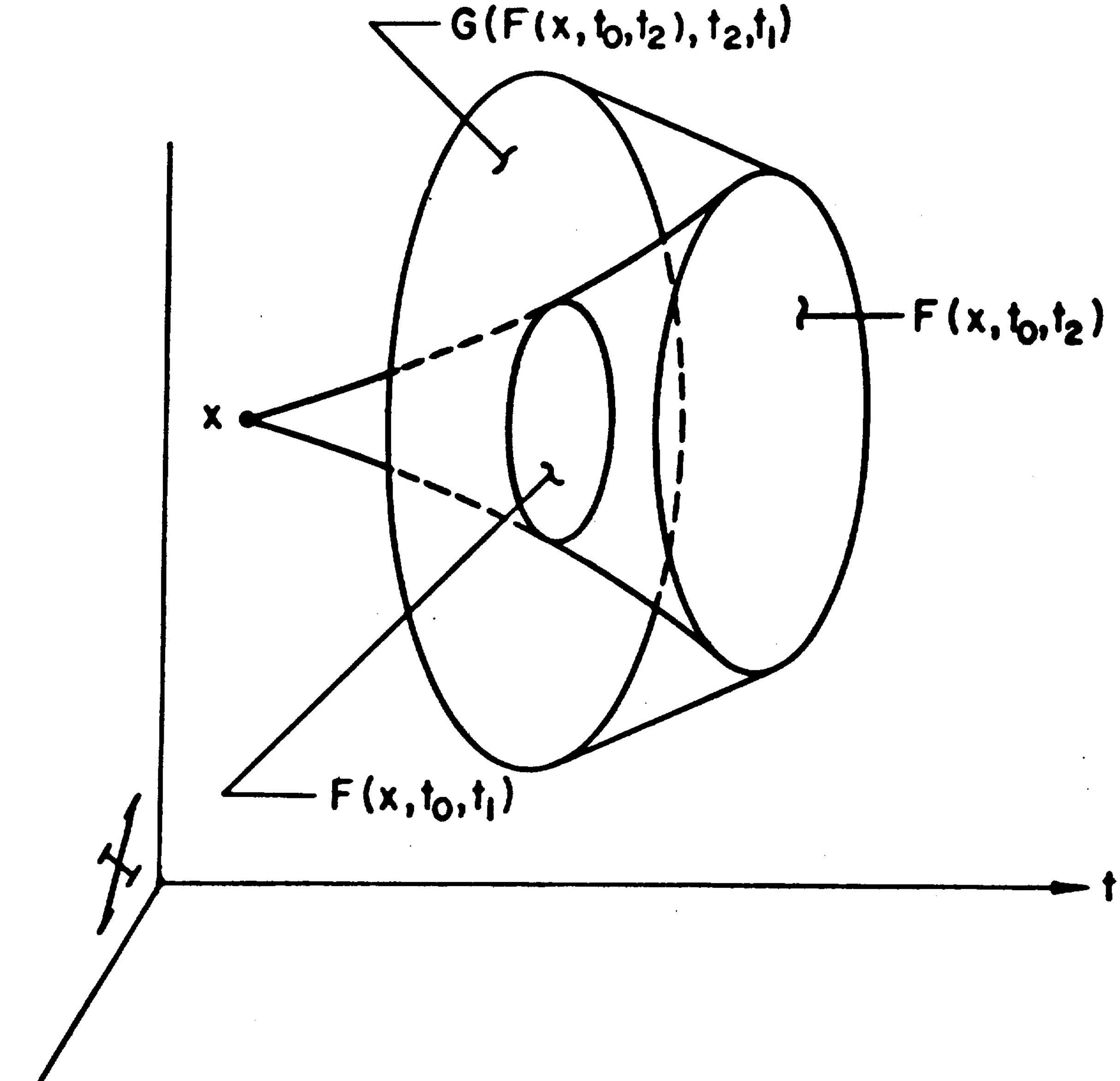

Fig. 7

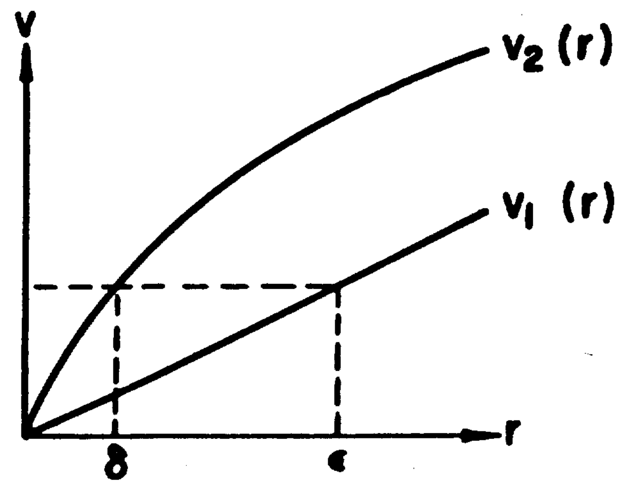

Fig. 8